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Global dynamics of a Leslie host-parasite model

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ABSTRACT

We consider the system of difference equations

$$x_{n+1}=\frac{\alpha\, x_n}{1+\beta\, y_n},\quad y_{n+1}=\frac{\gamma\, x_n\, y_n}{x_n+\delta\, y_n},\quad n=0,1,2,\ldots,$$

where $\alpha,\beta,\gamma,\delta,x_0,y_0$ are positive real numbers. This system was formulated by P. H. Leslie in 1948 and the present manuscript provides the most complete dynamical analysis to date. A boundedness and persistence result along with global attractivity results for various parameter regions are established.

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1. Introduction

A host-parasitoid model is a type of prey-predator model where the development of the attacking species (parasitoid) depends on the quantity of the food species (host) made available to it and the population of the food species depends on how many of its peers survived the infestation [1,4]. Host-parasite models have a similar structure to that of host-parasitoid models with the biggest difference being that the parasite may not kill the host [1,3]. These models have attracted the attention of many authors in recent years and several interesting systems are studied in [2,5,13,16,17]. One host-parasite model of particular interest, formulated in 1948 by P. H. Leslie, is given by

$$N_1(t+1)=\frac{\lambda_1\, N_1(t)}{1+(\lambda_1-1)\,\frac{N_2(t)}{K_2}},\quad N_2(t+1)=\frac{\lambda_2\, N_2(t)}{1+(\lambda_2-1)\,\frac{K_1\, N_2(t)}{K_2\, N_1(t)}},\quad t=0,1,2,\ldots,\tag{1}$$

where $\lambda_1,\lambda_2>1$ and K_1,K_2 are positive constants (see [11, p. 239]). The quantity N_1 represents the population of the host and N_2 represents the population of the parasite. An increase in the parasite population N_2 results in a decrease in the host population N_1 and an increase in the ratio $\frac{N_2}{N_1}$ results in a decrease in the parasite population as they lack resources to survive. See [11,12] for more information on (1). System (1) can be rewritten as

$$x_{n+1}=\frac{\alpha\, x_n}{1+\beta\, y_n},\quad y_{n+1}=\frac{\gamma\, x_n\, y_n}{x_n+\delta\, y_n},\quad n=0,1,2,\ldots,\tag{2}$$

where $\alpha,\beta,\gamma,\delta,x_0,y_0$ are positive real numbers.

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System (2) has been studied by Q. Din and T. Donchev, who claim in Theorem 6 of [3] that when $\alpha, \gamma > 1$ the unique positive equilibrium is a global attractor. However, the proof in [3] is incorrect as we now explain. The analysis of system (2) in [3] relies on Theorem 5 of [3], which is a result that appeared first as Theorems 2.2.9 and 2.2.11 in the PhD thesis of M. Nurkanović [14]. Also see [10]. A generalization of these results is Theorem 3 in [9]. The result of Nurkanović (or Theorem 3 of [9]) guarantees boundedness and persistence of solutions to (2) on sets $[m_1, M_1] \times [m_2, M_2]$ that are invariant under the map associated with the system. The purported proof of Theorem 6 in [3] failed to verify that nontrivial invariant sets $[m_1, M_1] \times [m_2, M_2]$ exist, and therefore global attractivity of the equilibrium was not established. In fact, no such sets exist: if $[m_1, M_1] \times [m_2, M_2]$ is an invariant subset of the positive quadrant of the plane, then by monotonicity and invariance,

$$m_1 \le \frac{\alpha\, m_1}{1+\beta\, M_2} \quad \text{and} \quad \frac{\alpha\, M_1}{1+\beta\, m_2} \le M_1\,. \tag{3}$$

From (3), one obtains $1+\beta\, M_2 \le \alpha$ and $\alpha \le 1+\beta\, m_2$, hence $m_2 = M_2$. A similar calculation gives $m_1 = M_1$, and it follows that the invariant set consists of just one point. Consequently, Nurkanović's result cannot be used to prove that the positive equilibrium in (2) is a global attractor. The present manuscript provides a proof, among other things, of the global attractivity of the unique positive equilibrium as well as the boundedness and persistence of solutions to system (2) under certain parameter restrictions that include those considered by Din and Donchev. The results in the coming sections provide the most complete analysis to date of model (1) formulated by P. H. Leslie in 1948.

Before we state the main result of this paper, it is convenient to introduce the change of variables

$$x' = \frac{\beta}{\delta}\, x\,, \quad y' = \beta\, y.$$

This change of variables allows for the elimination of the parameters β and δ, and after renaming variables, system (2) is transformed to

$$x_{n+1} = \frac{\alpha\, x_n}{1+y_n}, \quad y_{n+1} = \frac{\gamma\, x_n\, y_n}{x_n + y_n}, \quad n = 0, 1, 2, \dots. \tag{4}$$

An elementary calculation gives that a positive equilibrium for (4) exists if and only if $\alpha > 1$ and $\gamma > 1$. When this equilibrium exists, it is unique and given by

$$(\bar{x}_+\,, \bar{y}_+) := \left(\frac{\alpha - 1}{\gamma - 1}\,, \alpha - 1 \right). \tag{5}$$

Furthermore, if the positive equilibrium (5) exists, it is locally asymptotically stable [3].

The main result of this paper is Theorem 1, which is presented below.

Theorem 1: *Assume α, γ are arbitrary positive real numbers. Then system (4) has a positive equilibrium $(\bar{x}_+\,, \bar{y}_+)$ if and only if $\alpha > 1$ and $\gamma > 1$. If it exists, the positive equilibrium is unique and given by (5). For arbitrary positive numbers x_0 and y_0, let $\{(x_n, y_n)\}$ be given by (4). Then the following statements are true:*

(i) *If $\alpha < 1$, then $(x_n, y_n) \to (0, 0)$.*
(ii) *If $\alpha = 1$ and $\gamma < 1$, then $y_n \to 0$ and there exists $\bar{x} \ge 0$ such that $\{x_n\}$ is monotonically decreasing and converges to $\bar{x}$.*

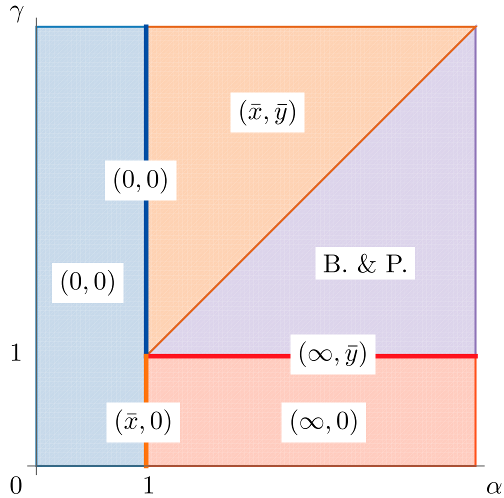

Figure 1. Parameter space regions. Here 'B. & P.' stands for *bounded and persistent orbits*, $(\bar{x}, \bar{y})$ stands for *orbits converge to a unique positive equilibrium*, $(\bar{x}, 0)$ stands for *orbits converge to a point on the x-axis*, and so on.

(iii) *If* $\alpha = 1$ *and* $\gamma \geq 1$, *then* $(x_n, y_n) \to (0, 0)$.
(iv) *If* $\alpha > 1$ *and* $\gamma < 1$, *then* $x_n \to \infty$ *and* $y_n \to 0$.
(v) *If* $\alpha > 1$ *and* $\gamma = 1$, *then* $x_n \to \infty$ *and, for some* $\bar{y} \geq 0$, $y_n \to \bar{y}$.
(vi) *If* $1 < \alpha \leq \gamma$, *then* $(x_n, y_n) \to (\bar{x}_+, \bar{y}_+)$. *Furthermore, the positive equilibrium* $(\bar{x}_+, \bar{y}_+)$ *is globally asymptotically stable on* $(0, \infty) \times (0, \infty)$.
(vii) *If* $1 < \gamma < \alpha$, *the sequence* $\{(x_n, y_n)\}$ *is bounded and persistent in* $(0, \infty) \times (0, \infty)$. *Also, for some choices of* γ, α *there exist nontrivial periodic solutions.*

The seven dynamical scenarios described in Theorem 1 are depicted in Figure 1 where the various parameter regions are labelled according to the long-term behaviour of solutions $\{(x_n, y_n)\}$ of system (4).

Our proof of Theorem 1 can be described as consisting of three main parts. The first part comprises proofs of statements (i) through (v) regarding the global behaviour of solutions to (4) in the absence of a positive equilibrium. This is done with elementary arguments in Section 2.

The second part of our proof of Theorem 1, presented in Section 3, treats statement (vi) regarding the global attractivity of the positive equilibrium when $1 < \alpha \leq \gamma$. A well known approach to proving global asymptotic stability of an equilibrium (hence global attractivity) is to establish the existence of a nonnegative function $L(x, y)$ that serves as a global, strict Lyapunov function for the map associated with the system (Theorem 2.16 in [6]). Due to the local asymptotic stability property of the equilibrium in our case, it is enough to have that $L(x, y)$ is a global strict Lyapunov function for the *second* iterate of the map. To establish this, it is sufficient to verify $\{(x, y) : L(x, y) = 0\}$ is a singleton set consisting of the equilibrium point, and that for $c > 0$, the *sub-level sets* $\{(x, y) : L(x, y) \leq c\}$ satisfy certain properties. Specifically, the sub-level sets must be a family of compact invariant neighborhoods of the positive equilibrium that cover the positive quadrant and satisfy the property that each such set is mapped into its interior under the second iterate of the map. For this purpose, we perform a transformation that makes $(1, 1)$ the unique equilibrium

Table 1. Some periodic points of system (4) for $\alpha = 1000$, $\gamma = 4$. These (approximate) periodic points were found numerically, and they can be calculated to any desired number of significant digits with a computer algebra system such as Mathematica by using extended precision arithmetic and taking advantage of the fact that the parameter values are natural numbers.

Point	Period
(7.23296542408588, 119.37344586002276)	9
(2.9537477933807694, 74.9804098699995)	10
(1.315953175890067, 44.70579950089623)	11
(0.6162568330798077, 24.574047842527435)	12
(0.30216562897564875, 11.571433168943637)	13

point. Then we prove that the sub-level sets of the *Kobayashi internal metric* on the positive cone (see [8, p. 86, Lemma 3.3.5 (iv)]), parametrized as

$$\mathcal{P}_\mu := \{\, (x,y) \;:\; \tfrac{1}{\mu} \le x \le \mu \,,\; \tfrac{1}{\mu} \le y \le \mu \,,\; \tfrac{1}{\mu}\, x \le y \le \mu\, x \,\}, \quad \mu \ge 1,$$

satisfy the desired properties (see Proposition 1 and Corollary 1). We do this without any further explicit reference to Lyapunov Theory. We are thus able to establish global attractivity of the positive equilibrium point when $1 < \alpha \le \gamma$.

The third part of our proof of Theorem 1, which concerns the remaining parameter region $1 < \gamma < \alpha$, is presented in Section 4. We found that periodic solutions exist for many parameter selections satisfying $1 < \gamma < \alpha$. For example, with $\alpha = 1000$, $\gamma = 4$, the unique equilibrium point (333, 999) coexists with several nontrivial periodic points, shown in Table 1. Therefore global attractivity of the unique equilibrium does not necessarily hold for parameters[1] α, γ satisfying $1 < \gamma < \alpha$. However, the existence of periodic solutions does not preclude the system from having other global properties. Indeed we establish boundedness and persistence of solutions when $1 < \gamma < \alpha$ (i.e. statement (vii) of Theorem 1) in Section 4. This is done as follows. A solution $\{(x_n, y_n)\}$ to (4) is *persistent* if there exists $\delta > 0$ such that $\min\,(x_n, y_n) \ge \delta$ for all $n \ge 0$. A logarithmic change of coordinates is performed to make the phase space the whole plane, so that, per Proposition 2, the question of boundedness and persistence of solutions to (4) is reduced to the question of boundedness of solutions in the new coordinates. A cover of the plane by compact sets $\mathcal{K}_\tau$ with $\tau > 0$ is constructed, such that for large enough τ, $\mathcal{K}_\tau$ is invariant under the map $\hat{T}$ associated with the system in logarithmic coordinates. The map $\hat{T}$ is given in Equation (22), and the sets $\mathcal{K}_\tau$ are introduced in Definition 1. Boundedness of solutions follows from the existence of the sets $\mathcal{K}_\tau$ and their corresponding properties. See Figure 2 for a visual representation of this idea.

The sets $\mathcal{K}_\tau$, introduced in Section 4 of this paper, are constructed with the help of certain auxiliary maps which are obtained from the main map of the system by applying suitable changes (see the comments after Corollary 2). This approach may be applicable to more general planar difference equations in order to prove boundedness of solutions. To our knowledge, this is the first time that it has been implemented.

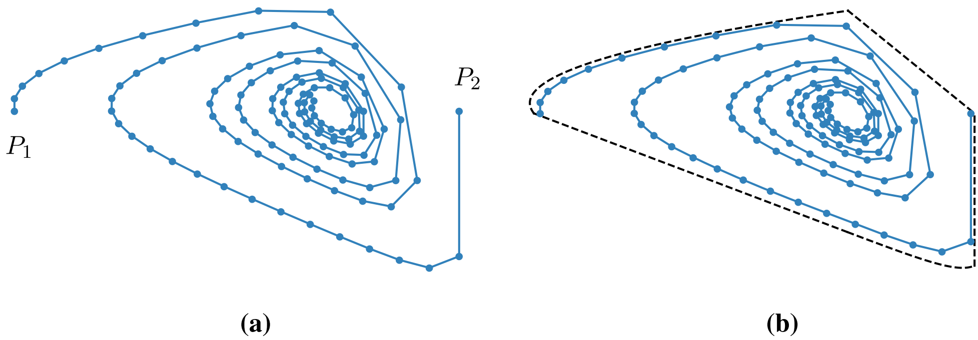

Figure 2. (a) The first few points of two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$, having initial points P_1 and P_2. (b) A way to prove that the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are bounded is to find a compact set $\mathcal{K}$ (outlined) which is invariant under the map $\hat{T}$, and that contains the initial point of the orbits. By invariance, $\mathcal{K}$ contains all the points of the orbit.

2. Global behaviour in the absence of a positive equilibrium

This section presents a proof of statements (i) through (v) of Theorem 1. Suppose first that $\alpha \le 1$. Choose arbitrary positive real numbers x_0 and y_0. With (x_n, y_n) given by (4) for $n > 0$, we have

$$x_{n+1} = \frac{\alpha\, x_n}{1 + y_n} < \alpha\, x_n \le x_n \quad \text{for all} \quad n \ge 0, \tag{6}$$

and thus

$$x_n \text{ converges to a nonnegative real number } \bar{x}. \tag{7}$$

Assume $\alpha < 1$. If $\bar{x} > 0$, then $y_n \to \alpha - 1$ by (6), which is impossible for $\alpha < 1$. Therefore $x_n \to 0$ as $n \to \infty$. From (4), $y_n \to 0$ as $n \to \infty$ and statement (i) follows.

If $\alpha = 1$, then there is a continuum of equilibrium points on the extended domain $(0, \infty) \times [0, \infty)$, consisting of points of the form $(x, 0)$ where $x > 0$.

With $\alpha = 1$, we consider three cases: $\gamma < 1$, $\gamma = 1$, and $\gamma > 1$.

$\alpha = 1, \gamma < 1$: From (4), $y_{n+1} < \gamma\, y_n$, thus $\{y_n\}$ is decreasing and convergent to some $\bar{y} \ge 0$. If $\bar{y} > 0$, then from (4),

$$x_n = \frac{y_{n+1}\, y_n}{\gamma\, y_n - y_{n+1}} \to \frac{\bar{y}^2}{\gamma\, \bar{y} - \bar{y}} = \frac{\bar{y}}{\gamma - 1}. \tag{8}$$

The last term in (8) is negative. This implies that $\bar{y} = 0$. From this and (7) we conclude $x_n \to \bar{x} > 0$ and $y_n \to 0$. Statement (ii) follows.

$\alpha = 1, \gamma = 1$: We have $y_{n+1} = \frac{x_n\, y_n}{x_n + y_n} < y_n$, therefore $\{y_n\}$ is a decreasing sequence that converges to some $\bar{y} \ge 0$. From $x_{n+1}(1 + y_n) = x_n$ we have $\bar{x}\,(1 + \bar{y}) = \bar{x}$. It follows that if $\bar{x} > 0$, then $\bar{y} = 0$. But if $\bar{x} = 0$, $y_{n+1}(x_n + y_n) = x_n\, y_n$ implies $\bar{y}\,(0 + \bar{y}) = 0\,\bar{y} = 0$, that is, $\bar{y} = 0$. Therefore $(x_n, y_n) \to (\bar{x}, 0)$ for some $\bar{x} \ge 0$. We claim that $\bar{x} = 0$. Suppose $\bar{x} > 0$. Consider the map R associated with (4) when $\alpha = \gamma = 1$:

$$R(x, y) := \left(\frac{x}{1 + y}, \frac{x\, y}{x + y} \right), \quad (x, y) \in (0, \infty) \times (0, \infty).$$

The map R has a real analytic extension $\tilde{R}$ to a neighborhood $\mathcal{N} \subset \mathbb{R}^2$ of $(\bar{x}, 0)$. It is shown in [7] that if $\mathcal{N}$ is small enough, then for every point $(x, y) \in \mathcal{N} \setminus \{(\bar{x}, 0)\}$ there exists $n > 0$ such that $\tilde{R}^n(x, y) \notin \mathcal{N}$. This contradicts $x_n \to \bar{x}$, so $\bar{x} = 0$. We conclude $x_n \to 0$ and $y_n \to 0$.

$\alpha = 1, \gamma > 1$: We claim $\bar{x} = 0$. Suppose this is not the case, i.e. $\bar{x} > 0$. Then $1 + y_n = \frac{x_{n+1}}{x_n} \to 1$, so $y_n \to 0$. Also,

$$\frac{y_{n+1}}{y_n} = \frac{\gamma\, x_n}{x_n + y_n} \to \gamma > 1,$$

which implies $y_n \not\to 0$. This contradicts the assumption, hence $x_n \to 0$. We have,

$$y_{n+2} = \frac{\gamma\, x_{n+1}\, y_{n+1}}{x_{n+1} + y_{n+1}} = \frac{\gamma\, x_n\, y_{n+1}}{(1 + y_n)(x_{n+1} + y_{n+1})} < \frac{\gamma\, x_n}{1 + y_n} < \gamma\, x_n\,.$$

Therefore, $y_n \to 0$ and statement (iii) follows.

Now, suppose that $\alpha > 1$ and $\gamma < 1$. Using system (4),

$$y_{n+1} = \frac{\gamma\, x_n\, y_n}{x_n + y_n} < \frac{\gamma\, x_n\, y_n}{x_n} = \gamma\, y_n \quad \text{for all} \quad n \geq 0,$$

and thus $y_n \to 0$ as $n \to \infty$. Furthermore, since $\alpha > 1$, there exists $N > 0$ and $A > 1$ such that $\frac{\alpha}{1+y_n} > A$ for all $n \geq N$. Then,

$$x_{n+1} = \frac{\alpha\, x_n}{1 + y_n} > A\, x_n\,, \quad n \geq N\,. \tag{9}$$

Consequently, $x_n \to \infty$ as $n \to \infty$ and statement (iv) follows.

If $\alpha > 1$ and $\gamma = 1$, we have

$$y_{n+1} = \frac{x_n\, y_n}{x_n + y_n} < y_n \quad \text{for all} \quad n \geq 0.$$

Thus there exists $\bar{y} \geq 0$ such that $y_n \downarrow \bar{y}$. If $\bar{y} = 0$, then from (9) we have $x_n \to \infty$. If $\bar{y} > 0$, then from (4), $x_n\,(y_n - y_{n+1}) = y_{n+1}\, y_n$ for $n \geq 0$. As $n \to \infty$, $y_{n+1}\, y_n \to \bar{y}^2 > 0$, which implies $x_n \to \infty$ and statement (v) follows. $\square$

3. Global attractivity of the positive equilibrium

This section provides a proof of statement (vi) of Theorem 1. Throughout the section we assume that $1 < \alpha \leq \gamma$. Under this assumption, there exists a unique positive equilibrium (5) for system (4). Furthermore, the change of variables

$$x' = \left(\frac{\alpha - 1}{\gamma - 1}\right)\frac{1}{x}, \quad y' = (\alpha - 1)\,\frac{1}{y}$$

conjugates system (4) to

$$x_{n+1} = a\, x_n + (1 - a)\,\frac{x_n}{y_n}, \quad y_{n+1} = (1 - b)\, x_n + b\, y_n, \quad n = 0, 1, 2, \ldots, \tag{10}$$

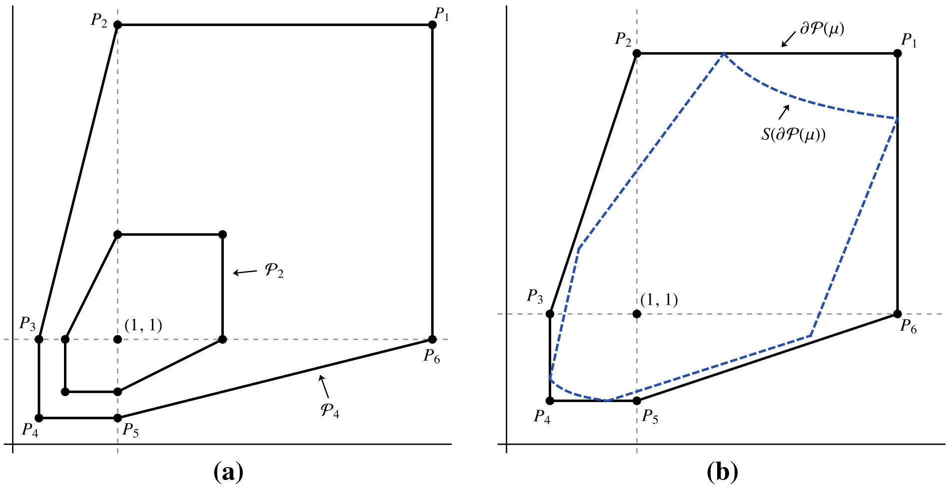

Figure 3. (a) The sets $\mathcal{P}_2$ and $\mathcal{P}_4$. (b) The boundary of a set $\mathcal{P}_\mu$ (solid) and its image $S(\partial\mathcal{P}_\mu)$ (dashed).

where the parameters a and b are

$$a = \frac{1}{\alpha} \quad \text{and} \quad b = \frac{1}{\gamma}. \tag{11}$$

The map associated with (10) on the positive quadrant is given by

$$S(x, y) = \left(a\,x + (1-a)\,\frac{x}{y}, (1-b)\,x + b\,y \right), \quad (x, y) \in (0, \infty) \times (0, \infty)\,. \tag{12}$$

The assumption $1 < \alpha \le \gamma$ is equivalent to

$$0 < b \le a < 1, \tag{13}$$

in which case the map S has a unique positive fixed point, namely $(1, 1)$. We shall prove statement (vi) of Theorem 1 by proving a similar result for (10) under assumption (13).

For $\mu \ge 1$, let

$$\mathcal{P}_\mu := \{\, (x, y)\ :\ \tfrac{1}{\mu} \le x \le \mu\,,\ \tfrac{1}{\mu} \le y \le \mu\,,\ \tfrac{1}{\mu}\,x \le y \le \mu\,x \,\}\,.$$

The sets $\mathcal{P}_2$ and $\mathcal{P}_4$ can be seen in Figure 3. Note that for each $\mu > 1$, the set $\mathcal{P}_\mu$ is the convex hull of the points

$$P_1 = (\mu, \mu),\ P_2 = (1, \mu),\ P_3 = (\tfrac{1}{\mu}, 1),\ P_4 = (\tfrac{1}{\mu}, \tfrac{1}{\mu}),\ P_5 = (1, \tfrac{1}{\mu}),\ P_6 = (\mu, 1)\,. \tag{14}$$

Some properties of the sets $\mathcal{P}_\mu$ are given in Proposition 1 below.

Proposition 1: *The following statements are true:*

(i) $(1, 1) \in \mathcal{P}_\mu$ *for each* $\mu > 1$, *and* $\mathcal{P}_1 = \{(1, 1)\}$.
(ii) $\cup\{P_\mu : \mu > 1\} = (0, \infty) \times (0, \infty)$.

(iii) *For every* $(x, y) \neq (1, 1)$ *there exists* $\nu > 1$ *such that* $(x, y) \in \partial\mathcal{P}_\nu$.
(iv) $S(\mathcal{P}_\mu) \subset \mathcal{P}_\mu$ *for each* $\mu > 1$.
(v) $S^2(\mathcal{P}_\mu) \subset int(\mathcal{P}_\mu)$ *for each* $\mu > 1$.

Statements (i) through (iii) of Proposition 1 are obviously true, so here we only prove (iv) and (v). Before we do so, we state a corollary to Proposition 1 that is equivalent to statement (vi) of Theorem 1.

Corollary 1: *For every* $(x, y) \in (0, \infty) \times (0, \infty)$, $S^n(x, y) \to (1, 1)$.

Proof: If $(x, y) \in (0, \infty) \times (0, \infty)$, then (ii) of Proposition 1 implies $(x, y) \in \mathcal{P}_\mu$ for some $\mu > 1$. By (iv) of Proposition 1, $S^n(x, y) \in \mathcal{P}_\mu$ for all $n \geq 1$ and thus $\{S^n(x, y)\}$ has at least one accumulation point $(\bar{x}, \bar{y})$. Suppose $(\bar{x}, \bar{y}) \neq (1, 1)$, then by (iii) there exists $\nu > 1$ such that $(\bar{x}, \bar{y}) \in \partial\mathcal{P}_\nu$. By continuity of S and by (iv) and (v), $S^n(x, y) \in \text{int}(\mathcal{P}_\nu)$ for all n sufficiently large. This is not possible since $(\bar{x}, \bar{y})$ is an accumulation point of $\{S^n(x, y)\}$ and therefore $(\bar{x}, \bar{y}) = (1, 1)$. Since $(1, 1)$ is the only accumulation point of the bounded sequence $\{S^n(x, y)\}$, it follows that $S^n(x, y) \to (1, 1)$. □

Now, for the proof of (iv) and (v) of Proposition 1, let $\mu > 1$ be fixed but arbitrary, and let $P_1, \ldots, P_6$ be the extreme points or vertices of $\mathcal{P}_\mu$ given in (14). We claim first that

$$S(P_\ell) \in \mathcal{P}_\mu \quad \text{for } 1 \leq \ell \leq 6. \tag{15}$$

From (12) and (14),

$$\begin{aligned}
S(P_1) &= S(\mu, \mu) &&= (1 + a\,(\mu - 1), \mu) &&\in [P_1, P_2],\\
S(P_3) &= S\left(\tfrac{1}{\mu}, 1\right) &&= \left(\tfrac{1}{\mu}, \tfrac{1}{\mu} + b\left(\tfrac{\mu-1}{\mu}\right)\right) &&\in [P_3, P_4],\\
S(P_4) &= S\left(\tfrac{1}{\mu}, \tfrac{1}{\mu}\right) &&= \left(1 + a\left(\tfrac{1-\mu}{\mu}\right), \tfrac{1}{\mu}\right) &&\in [P_4, P_5],\\
S(P_6) &= S\left(\mu, 1\right) &&= (\mu, \mu + b\,(1 - \mu)) &&\in [P_6, P_1].
\end{aligned}$$

Furthermore, $S(P_2) = S(1, \mu) = (a + \frac{1-a}{\mu}, 1 + b\,(\mu - 1))$ and it can be readily seen that the following inequalities are true:

$$\tfrac{1}{\mu} \leq a + \tfrac{1-a}{\mu} \leq \mu\,,\ \tfrac{1}{\mu} \leq 1 + b(\mu - 1) \leq \mu\,,\ \tfrac{1}{\mu}\,(a + \tfrac{1-a}{\mu}) \leq 1 + b(\mu - 1) \leq \mu\,(a + \tfrac{1-a}{\mu}).$$

That is, $S(P_2) \in \mathcal{P}_\mu$. Finally,

$$S(P_5) = S\left(1, \tfrac{1}{\mu}\right) = \left(a\,(1 - \mu) + \mu, b\left(\tfrac{1-\mu}{\mu}\right) + 1\right),$$

and one can similarly conclude that $S(P_5) \in \mathcal{P}_\mu$. Thus (15) has been established.

To prove (iv), it is sufficient to prove $S(\partial\mathcal{P}_\mu) \subset \mathcal{P}_\mu$. We have

$$\begin{aligned}
S([P_1, P_2]) = \Bigg\{\Bigg(&\frac{(a\,(\mu - 1) + 1)((\mu - 1)\,t + 1)}{\mu},\\
&b\,(\mu - \mu\,t + t - 1) + (\mu - 1)\,t + 1\Bigg)\ :\ 0 \leq t \leq 1\Bigg\}.
\end{aligned}$$

Hence $S([P_1, P_2])$ is a line segment with endpoints in the set $\mathcal{P}_\mu$, which is convex. Therefore, $S([P_1, P_2]) \subset \mathcal{P}_\mu$. Similar considerations lead to $S([P_2, P_3]) \subset \mathcal{P}_\mu$, $S([P_4, P_5]) \subset \mathcal{P}_\mu$, and $S([P_5, P_6]) \subset \mathcal{P}_\mu$.

For $0 \leq t \leq 1$, let $x(t)$ and $y(t)$ be defined by the equation

$$(x(t), y(t)) := S\big((1-t)\,P_3 + t\,P_4\big) = \left(\frac{\mu + at - a\,\mu\,t}{\mu\,(\mu + t - \mu\,t)}, \frac{1 + b\,(-1 + \mu + t - \mu\,t)}{\mu}\right).$$

Then $S([P_3, P_4]) = \{(x(t), y(t)) : 0 \leq t \leq 1\}$, and for $0 \leq t \leq 1$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(1-a)(\mu-1)}{(\mu + t - \mu\,t)^2} > 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{b\,(1-\mu)}{\mu} < 0. \tag{16}$$

Hence, from (15) and (16), $S([P_3, P_4]) \subset [\frac{1}{\mu}, 1] \times [\frac{1}{\mu}, 1]$ and we conclude $S([P_3, P_4]) \subset \mathcal{P}_\mu$. A similar proof (omitted here) yields $S([P_6, P_1]) \subset \mathcal{P}_\mu$. This completes the proof of (iv).

For part (v), from the proof of part (iv), if $(x, y) \in \mathcal{P}_\mu$, then $S(x, y) \in \partial\mathcal{P}_\mu$ only when $(x, y) \in \{P_1, P_3, P_4, P_6\}$, and otherwise $S(x, y) \in \mathrm{int}(\mathcal{P}_\mu)$ and $S^2(x, y) \in \mathrm{int}(\mathcal{P}_\mu)$. In addition, for $1 \leq \ell \leq 6$, $S(P_\ell) \notin \{P_1, P_3, P_4, P_6\}$, so $S^2(P_\ell) \in \mathrm{int}(\mathcal{P}_\mu)$. It follows that $S^2(\mathcal{P}_\mu) \subset \mathrm{int}(\mathcal{P}_\mu)$. □

4. Boundedness and persistence of solutions

A proof of boundedness and persistence of solutions of system (4) for $1 < \gamma < \alpha$ is presented in this section, which corresponds to the first part of statement (vii) of Theorem 1. The second part of (vii) concerning the existence of nontrivial periodic points is justified in the Introduction (Section 1), see Table 1 and associated comments.

4.1. Structure of the Proof of Statement (vii) of Theorem 1

Throughout the section we shall assume the inequality

$$1 < \gamma < \alpha. \tag{17}$$

Under this assumption, there exists a unique positive equilibrium (5) for system (4). The change of variables

$$x' = \left(\frac{\alpha - 1}{\gamma - 1}\right)\frac{1}{x}, \quad y' = \left(\frac{1}{\alpha - 1}\right)y$$

conjugates system (4) to

$$x_{n+1} = a\,x_n + (1-a)\,x_n\,y_n, \quad y_{n+1} = \frac{y_n}{(1-b)\,x_n\,y_n + b}, \qquad n = 0, 1, 2, \ldots, \tag{18}$$

where a and b are as in (11). The map corresponding to (18) is given by

$$T(x, y) = \left(a\,x + (1-a)\,x\,y, \frac{y}{(1-b)\,x\,y + b}\right), \quad (x, y) \in (0, \infty) \times (0, \infty). \tag{19}$$

Assumption (17) becomes

$$0 < a < b < 1, \tag{20}$$

in which case the map T has a unique positive fixed point, namely $(1, 1)$. We shall prove (vii) of Theorem 1 by proving a similar statement for (18) under assumption (20).

It is useful to consider logarithmic coordinates. Denote with L and E the planar maps defined for $(x, y) \in (0, \infty) \times (0, \infty)$ and $(u, v) \in \mathbb{R}^2$, respectively by

$$L(x, y) := (\ln(x), \ln(y)) \quad \text{and} \quad E(u, v) := (e^u, e^v). \tag{21}$$

Set $\hat{T} := L \circ T \circ E$. That is,

$$\hat{T}(u, v) = \left(\ln(a\, e^u + (1-a)\, e^{u+v}), \ln\left(\frac{e^v}{(1-b)\, e^{u+v} + b} \right) \right), \quad (u, v) \in \mathbb{R}^2. \tag{22}$$

Thus $\hat{T}$ is a conjugate of T for which the origin is the (unique) fixed point. An immediate consequence of the definition of $\hat{T}$ is Proposition 2 presented below.

Proposition 2: *Let (x, y) be an arbitrary element of $(0, \infty) \times (0, \infty)$. Then the sequence $\{T^n(x, y)\}$ is bounded and persists in $(0, \infty) \times (0, \infty)$ if and only if $\{\hat{T}^n(L(x, y))\}$ is bounded in $\mathbb{R}^2$.*

It can also be shown that bounded subsets of $\mathbb{R}^2$ are contained in $\hat{T}$-invariant compact sets, as described in Proposition 3.

Proposition 3: *Suppose $0 < a < b < 1$. Then for any bounded set $\mathcal{B} \subset \mathbb{R}^2$ there exists a $\hat{T}$-invariant compact set $\mathcal{K}$ such that $\mathcal{B} \subset \mathcal{K}$.*

Propositions 2 and 3 have the following corollary, which is precisely statement (vii) of Theorem 1.

Corollary 2: *Let (x, y) be an arbitrary element of $(0, \infty) \times (0, \infty)$. Then the sequence $\{T^n(x, y)\}$ is bounded and persists in $(0, \infty) \times (0, \infty)$.*

The remainder of this section is devoted to proving Proposition 3. The proof involves constructing a family of compact sets $\mathcal{K}_\tau$ that satisfy the properties set forth in Proposition 3 for τ taken to be sufficiently large. Figure 4 shows a typical set $\mathcal{K}_\tau$. The boundary consists of five curves, three of which $\hat{D'}_0$, $\hat{D'}_2$, and $\hat{D'}_4$ are linear segments. The remaining two curves $\hat{D'}_1$ and $\hat{D'}_3$ are derived from careful analysis of the map's behaviour on certain regions of the plane, away from the origin. Indeed, the curves $\hat{D'}_1$ and $\hat{D'}_3$ are subsets of invariant curves of the maps $\hat{M}$ and $\hat{N}$ given in Equation (23), which approximate asymptotically $\hat{T}$ on quadrants II and IV respectively. The maps $\hat{M}$ and $\hat{N}$ are obtained from $\hat{T}$ by removing terms that can be neglected in selected regions of the plane. The maps obtained in this way have invariant curves that can be found explicitly. These invariant curves are used in turn to define $\hat{D'}_1$ and $\hat{D'}_3$.

Before exploring these ideas rigorously, we first present basic results about T and $\hat{T}$ as well as results related to the two auxiliary maps, $\hat{M}$ and $\hat{N}$, that are useful in the construction of $\mathcal{K}_\tau$ and for the arguments that follow.

4.2. Ancillary properties and maps

If $F = (f_1, f_2)$ is a map on a planar region $\mathcal{R}$, the *equilibrium curves* of F are the sets $\{(x, y) \in \mathcal{R} : f_1(x, y) = x\}$ and $\{(x, y) \in \mathcal{R} : f_2(x, y) = y\}$. The equilibrium curves of the maps T and $\hat{T}$ given in (19) and (22) play a prominent role in our proof. Before we go any

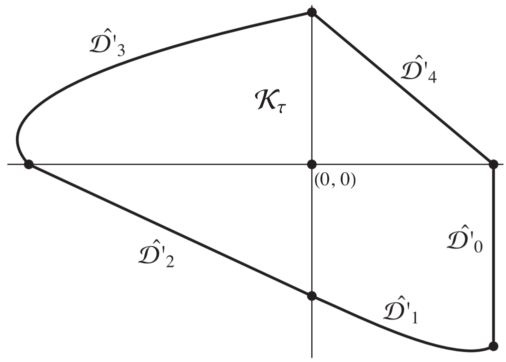

Figure 4. A set $\mathcal{K}_\tau$ whose boundary consists of the sets $\hat{\mathcal{D}}'_\ell$ for $0 \le \ell \le 4$.

further, we adopt the following convention in order to simplify notation use:

$$\textit{unless otherwise restricted}, \ (x, y) \in (0, \infty) \times (0, \infty) \text{ and } (u, v) \in \mathbb{R}^2.$$

The equilibrium curves of the maps T are as follows:

$$\mathcal{C}_1 := \{(x, y) : y = 1\} \quad \text{and} \quad \mathcal{C}_2 := \{(x, y) : x\,y = 1\}.$$

The equilibrium curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have $(1, 1)$ as their only common point, and the complement in the positive quadrant of their union consists of four disjoint connected components

$$\mathcal{R}_1 = \{(x, y) : y > 1 \text{ and } x\,y > 1\}, \mathcal{R}_2 = \{(x, y) : y > 1 \text{ and } x\,y < 1\},$$
$$\mathcal{R}_3 = \{(x, y) : y < 1 \text{ and } x\,y < 1\}, \mathcal{R}_4 = \{(x, y) : y < 1 \text{ and } x\,y > 1\}.$$

That is,

$$(0, \infty) \times (0, \infty) \setminus (\mathcal{C}_1 \cup \mathcal{C}_2) = \bigcup \{\mathcal{R}_\ell : 1 \le \ell \le 4\}.$$

Similarly, the equilibrium curves of the map $\hat{T}$ are

$$\hat{\mathcal{C}}_1 := \{(u, v) : v = 0\} \quad \text{and} \quad \hat{\mathcal{C}}_2 := \{(u, v) : u + v = 0\}\,.$$

The curves $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ have $(0, 0)$ as their only common point, and the complement in the plane of their union consists of four disjoint connected components

$$\hat{\mathcal{R}}_1 = \{(u, v) : v > 0 \text{ and } u + v > 0\}, \hat{\mathcal{R}}_2 = \{(u, v) : v > 0 \text{ and } u + v < 0\},$$
$$\hat{\mathcal{R}}_3 = \{(u, v) : v < 0 \text{ and } u + v < 0\}, \hat{\mathcal{R}}_4 = \{(u, v) : v < 0 \text{ and } u + v > 0\}.$$

That is,

$$\mathbb{R}^2 \setminus (\hat{\mathcal{C}}_1 \cup \hat{\mathcal{C}}_2) = \bigcup \{\hat{\mathcal{R}}_\ell : 1 \le \ell \le 4\}.$$

The sets $\mathcal{R}_\ell$ and $\hat{\mathcal{R}}_\ell$, $1 \le \ell \le 4$, are depicted in Figure 5. Now, denote with $\preceq_{se}$ the *South-East* partial order on $\mathbb{R}^2$ whose nonnegative cone is the standard fourth quadrant $\{(u, v) : u \ge 0, v \le 0\}$. That is, $(u_1, v_1) \preceq_{se} (u_2, v_2)$ if and only if $u_1 \le u_2$ and $v_1 \ge v_2$.

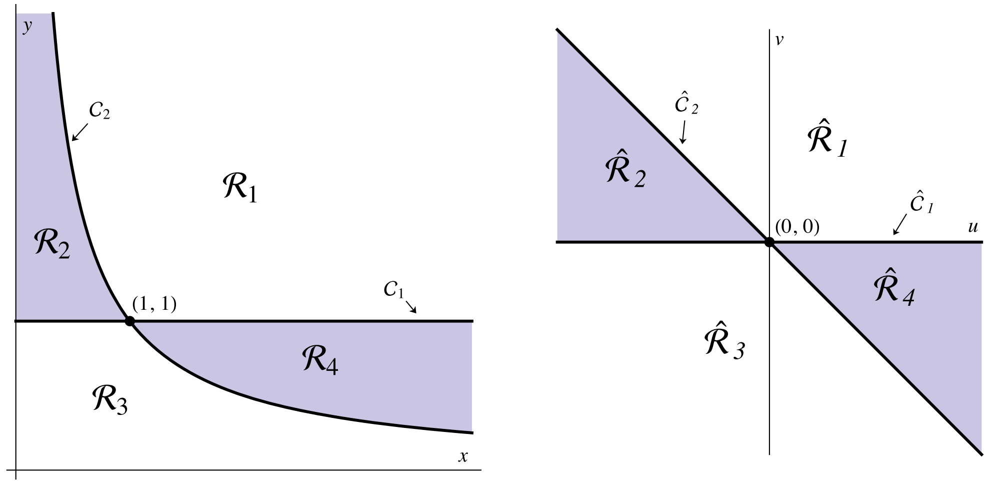

Figure 5. Equilibrium curves and complementary regions for T and $\hat{T}$, respectively.

Similarly, denote with $\preceq_{ne}$ the *North-East* partial order on $\mathbb{R}^2$ whose nonnegative cone is the standard first quadrant $\{(u, v) : u, v \geq 0\}$. That is, $(u_1, v_1) \preceq_{ne} (u_2, v_2)$ if and only if $u_1 \leq u_2$ and $v_1 \leq v_2$ (see [15]). Basic monotonicity properties can then be used to prove Proposition 4.

Proposition 4: *The following statements are true:*

(i) $(x, y) \preceq_{se} T(x, y)$ *for* $(x, y) \in \mathcal{R}_1$ (i′) $(u, v) \preceq_{se} \hat{T}(u, v)$ *for* $(u, v) \in \hat{\mathcal{R}}_1$
(ii) $(x, y) \preceq_{ne} T(x, y)$ *for* $(x, y) \in \mathcal{R}_2$ (ii′) $(u, v) \preceq_{ne} \hat{T}(u, v)$ *for* $(u, v) \in \hat{\mathcal{R}}_2$
(iii) $T(x, y) \preceq_{se} (x, y)$ *for* $(x, y) \in \mathcal{R}_3$ (iii′) $\hat{T}(u, v) \preceq_{se} (u, v)$ *for* $(u, v) \in \hat{\mathcal{R}}_3$
(iv) $T(x, y) \preceq_{ne} (x, y)$ *for* $(x, y) \in \mathcal{R}_4$ (iv′) $\hat{T}(u, v) \preceq_{ne} (u, v)$ *for* $(u, v) \in \hat{\mathcal{R}}_4$

We shall need the maps

$$M(x, y) := \left(a\,x\,,\ \frac{y}{(1-b)\,x\,y + b} \right), \quad (x, y) \in (0, \infty) \times (0, \infty),$$

and

$$N(x, y) := \left((1-a)\,x\,y\,,\ \frac{1}{b}\,y \right), \quad (x, y) \in (0, \infty) \times (0, \infty),$$

along with the corresponding conjugate maps $\hat{M}$ and $\hat{N}$ on $\mathbb{R}^2$ given in terms of the maps from (21) by

$$\hat{M} := L \circ M \circ E \quad \text{and} \quad \hat{N} := L \circ N \circ E. \tag{23}$$

To prove the boundedness and persistence of the solutions of system (18), it is important to understand the behaviour of the solutions for small values of x and y. Close inspection of the map in (19) reveals that T behaves similarly to the map M for values of y close to zero and T behaves similarly to the map N for values of x close to zero. In this way, M and N offer valuable insight into the behaviour of solutions of system (18). In Lemmas 1 and 3 that follow, it is proven that there exist invariant curves for the maps $\hat{M}$ and $\hat{N}$ in $\hat{\mathcal{R}}_2$ and $\hat{\mathcal{R}}_3$, respectively. These curves have important properties when related to the map $\hat{T}$

and play a role in the definition of the family of compact sets $\mathcal{K}_\tau$ needed for the proof of Proposition 3. Lemma 2 gives a property of the image of certain line segments in $\hat{\mathcal{R}}_3$. This is useful when proving the invariance of the sets $\mathcal{K}_\tau$ that are constructed.

We shall need the constant r given by

$$r := \frac{\ln(b)}{\ln(a)}. \tag{24}$$

Under assumption (20), r satisfies

$$0 < r < 1. \tag{25}$$

Lemma 1, given below, details an invariant curve corresponding to the map $\hat{M}$ along with properties of its image under $\hat{T}$.

Lemma 1: *Let τ be a fixed but otherwise arbitrary positive real number. Let $\hat{f}_1 : (-\infty, \tau] \to \mathbb{R}$ be the function given by*

$$\hat{f}_1(u) = -\ln\left(e^\tau\left(e^{r(u-\tau)} + \tfrac{1-b}{b-a}\left(e^{r(u-\tau)} - e^{u-\tau}\right)\right)\right). \tag{26}$$

and let $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}'_1$ be the sets

$$\hat{\mathcal{D}}_1 := \left\{(u,v) \in \mathbb{R}^2 : v = \hat{f}_1(u),\ u \le \tau\right\},$$

$$\hat{\mathcal{D}}'_1 := \left\{(u,v) \in \mathbb{R}^2 : v = \hat{f}_1(u),\ 0 \le u \le \tau,\ v \le 0\right\}.$$

Then $\hat{f}_1(\cdot)$ is a convex smooth function,

$$\hat{M}(\hat{\mathcal{D}}_1) \subset \hat{\mathcal{D}}_1, \quad and \quad \hat{T}(\hat{\mathcal{D}}'_1) \subset \left\{(u,v) \in \mathbb{R}^2 : \hat{f}_1(u) < v < 0,\ u < \tau\right\}. \tag{27}$$

Figure 6 shows the curve $\hat{\mathcal{D}}'_1$ described in Lemma 1 along with its image under the map $\hat{T}$. An extension of $\hat{\mathcal{D}}'_1$ and its corresponding image in the third quadrant are also included to illustrate the relation $\hat{T}(\hat{\mathcal{D}}'_1) \subset \left\{(u,v) \in \mathbb{R}^2 : \hat{f}_1(u) < v < 0,\ u < \tau\right\}$, which is needed in the arguments used in Section 4.4.

***Proof*:** A straightforward calculation gives

$$\hat{f}_1''(u) = \frac{(1-a)(1-b)(r-1)^2 e^{-r\tau+ru+\tau+u}}{\left((1-b)e^u - (1-a)e^{r(u-\tau)+\tau}\right)^2},$$

so $\hat{f}_1''(u)$ is well defined and positive for $u \le \tau + \frac{1}{r-1}\ln\left(\frac{1-b}{1-a}\right)$. This inequality, together with (20) and (25), imply that $\hat{f}_1''(u)$ is defined for $u \le \tau$, and consequently v is a convex function of u for $u \le \tau$.

With the change of coordinates $x = e^u$, $y = e^v$, together with $x_0 := e^\tau$ and

$$\mathcal{D}_1 := \left\{(x,y) : \tfrac{1}{x_0\, y} = \left(\tfrac{x}{x_0}\right)^r + \tfrac{1-b}{b-a}\left(\left(\tfrac{x}{x_0}\right)^r - \tfrac{x}{x_0}\right),\ x \le e^\tau\right\},$$

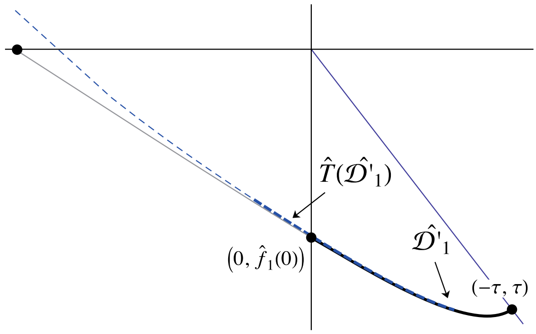

Figure 6. The curve $\hat{\mathcal{D}}_1'$ (thick, solid) and its image under $\hat{T}$ (thick, dashed) along with an extension of $\hat{\mathcal{D}}_1'$ (thin, solid) and its image under $\hat{T}$ (thin, dashed).

the inclusion $M(\mathcal{D}_1) \subset \mathcal{D}_1$ is equivalent to $\hat{M}(\hat{\mathcal{D}}_1) \subset \hat{\mathcal{D}}_1$. We prove the former. Suppose $(x, y) \in \mathcal{D}_1$, and set

$$(x', y') := M(x, y) = \left(a\,x, \frac{y}{(1-b)\,x\,y + b} \right).$$

Then $(x', y') \in \mathcal{D}_1$ if and only if $x' \le e^{\tau}$ and

$$\frac{(1-b)\,x\,y + b}{x_0\,y} = \left(\frac{a\,x}{x_0}\right)^r + \frac{1-b}{b-a}\left(\left(\frac{a\,x}{x_0}\right)^r - \frac{a\,x}{x_0}\right). \tag{28}$$

Through algebraic manipulation, Equation (28) may be rewritten as

$$\frac{b}{x_0\,y} = -(1-b)\,\frac{x}{x_0} + \left(\frac{a\,x}{x_0}\right)^r + \frac{1-b}{b-a}\left(\left(\frac{a\,x}{x_0}\right)^r - \frac{a\,x}{x_0}\right). \tag{29}$$

The equality $a^r = b$ and further simplification in (29) give the equation

$$\frac{1}{x_0\,y} = \left(\frac{x}{x_0}\right)^r + \frac{1-b}{b-a}\left(\frac{x}{x_0}\right)^r - \frac{1-b}{b-a}\left(\frac{x}{x_0}\right). \tag{30}$$

Since by assumption $(x, y) \in \mathcal{D}_1$, we have (30) is true. It is also the case that $x' = a\,x \le a\,e^{\tau} < e^{\tau}$. This proves $(x', y') \in \mathcal{D}_1$.

To prove the second inclusion in (27), consider $(u, v) \in \hat{\mathcal{D}}_1'$, and set $(u', v') = \hat{M}(u, v)$ and $(u'', v'') = \hat{T}(u, v)$. Thus $(u', v') \in \hat{\mathcal{D}}_1$. From the definition of $\hat{M}$ and $\hat{T}$ we have

$$u' < u'' \quad \text{and} \quad v' = v''. \tag{31}$$

Consider the function $\psi(t)$ with $t \le \tau$, given by

$$\psi(t) = e^{r(t-\tau)} + \frac{1-b}{b-a}\left(e^{r(t-\tau)} - e^{t-\tau}\right).$$

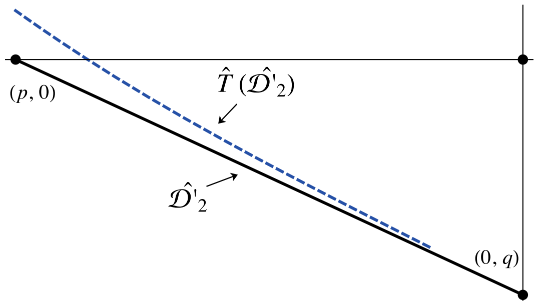

Figure 7. The curve $\hat{\mathcal{D}}_2'$ (solid) and its image under $\hat{T}$ (dashed).

Since $(u', v') \in \hat{\mathcal{D}}_1$ then $v' = \hat{f}_1(u') = -\ln(e^{\tau}\,\psi(u'))$. Therefore, $e^{-\tau-v'} = \psi(u')$. This fact, (31), and the increasing character of ψ give

$$e^{-\tau-v''} = e^{-\tau-v'} = \psi(u') < \psi(u'')\,. \tag{32}$$

Inequality (32) implies $\hat{f}_1(u'') < v''$, which together with

$$v'' = \ln\left(\frac{e^{v}}{(1-b)\,e^{u+v} + b}\right) < 0,$$

complete the proof of the second inclusion in (27). See Figure 6. □

Lemma 2, given below, details a property of the image under $\hat{T}$ of certain line segments.

Lemma 2: *Let p and q be arbitrary negative numbers such that $\frac{q}{p} < r$, where r is defined in (24). Let $\hat{\mathcal{D}}_2$ be the line in the plane through $(p, 0)$ and $(0, q)$, and let $\hat{\mathcal{D}}_2'$ be the line segment whose endpoints are $(p, 0)$ and $(0, q)$. Then $\hat{T}(\hat{\mathcal{D}}_2')$ is a subset of the connected component of $\mathbb{R}^2 \setminus \hat{\mathcal{D}}_2$ that contains the origin.*

***Proof*:** For $u \le 0$, $v < 0$, consider the real valued function

$$\phi(u, v) := -\frac{\ln((1-b)e^{u+v} + b)}{\ln(a + (1-a)e^{v})}.$$

We claim

$$\phi(u, v) \le -\frac{\ln(b)}{\ln(a)} = -r, \quad u \le 0,\ v < 0. \tag{33}$$

It can be easily shown that for fixed $v \le 0$, $\phi(u, v)$ is increasing in u. Therefore, it is sufficient to verify (33) for $\phi(0, v)$. Equivalently, with $y := e^{v}$, we will verify that $f(y) \le -r$ for all $y \in (0, 1)$, where

$$f(y) = -\frac{\ln((1-b)y + b)}{\ln(a + (1-a)y)}.$$

Notice,

$$t + \ln(1-t) < 0 \quad \text{for} \quad t \in (0, 1). \tag{34}$$

Therefore, for $r \in (0,1)$ and $y \in (0,1)$,

$$\frac{\partial}{\partial r}\left[\frac{((1-a^r)\,y+a^r)\ln((1-a^r)\,y+a^r)}{1-a^r}\right]$$
$$=\frac{a^r\,\ln(a)((1-a^r)(1-y)+\ln(1-(1-a^r)(1-y)))}{(1-a^r)^2}>0, \tag{35}$$

where (34) was used with $t=(1-a^r)(1-y)$ to conclude (35). The inequality in (35), along with $b=a^r$ from (24), imply

$$\frac{((1-b)\,y+b)\ln((1-b)\,y+b)}{1-b}=\frac{((1-a^r)\,y+a^r)\ln((1-a^r)\,y+a^r)}{1-a^r}$$
$$<\frac{((1-a)\,y+a)\ln((1-a)\,y+a)}{1-a}. \tag{36}$$

It follows from (36) that for all $y \in (0,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}y}[f(y)]=\frac{\frac{(1-a)\ln((1-b)y+b)}{(1-a)y+a}-\frac{(1-b)\ln((1-a)y+a)}{(1-b)y+b}}{(\ln((1-a)y+a))^2}<0.$$

Consequently, $f(y)\le f(0)=-\frac{\ln(b)}{\ln(a)}=-r$, and statement (33) is established. Now, assume $(u,v)\in\hat{\mathcal{D}}_2'$. Since

$$\hat{T}(u,v)-(u,v)=\left(\ln(a+(1-a)\,e^v),-\ln((1-b)\,e^{u+v}+b)\right), \tag{37}$$

the slope of the line through (u,v) and $\hat{T}(u,v)$ is precisely $\phi(u,v)$. By Proposition 4, $T(u,v)\preceq_{se}(u,v)$. From the latter relation, (33), (37), and the hypothesis on the slope of $\hat{\mathcal{D}}_2$, namely $-\frac{q}{p}$ being greater than $-r$, it follows that $\hat{T}(u,v)$ and $(0,0)$ belong to the same component of $\mathbb{R}^2\setminus\hat{\mathcal{D}}_2$. The curve $\hat{\mathcal{D}}_2'$ and its image under $\hat{T}$ can be seen in Figure 7. □

The final lemma in this subsection details an invariant curve corresponding to the map $\hat{N}$ along with properties of its image under $\hat{T}$. Prior to stating the lemma, we verify that

$$\hat{T}(\{(u,v):u+v\ge 0,\ u\le 0,\ v>0\})\subset\hat{\mathcal{R}}_1. \tag{38}$$

Consider $(u,v)\in\{(s,t):s+t\ge 0,\ s\le 0,\ t>0\}$ such that $u+v=0$ and notice

$$\hat{T}(u,v)=(\ln(a\,e^u+(1-a)),v). \tag{39}$$

Since $u<0$ implies $\ln(a\,e^u+(1-a))>u$, it follows from (39) that $\hat{T}(u,v)\in\hat{\mathcal{R}}_1$. Similarly, consider $(u,v)\in\{(s,t):s+t\ge 0,\ s\le 0,\ t>0\}$ such that $u=0$ and notice

$$\hat{T}(u,v)=\left(\ln(a+(1-a)e^v),\ln\left(\frac{e^v}{(1-b)e^v+b}\right)\right).$$

Since $v>0$, $\hat{T}(u,v)$ is in the first quadrant of the plane and thus belongs to $\hat{\mathcal{R}}_1$. By continuity of $\hat{T}$, (38) follows. This relation will be helpful in proving Lemma 3 below.

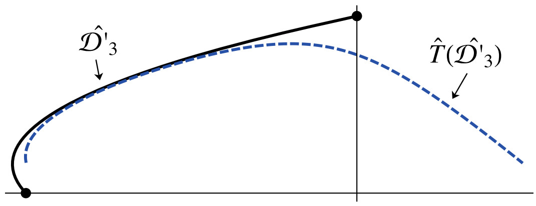

Figure 8. The curve $\hat{\mathcal{D}}'_3$ (solid) and its image under $\hat{T}$ (dashed).

Lemma 3: *Let c_0 be a fixed but otherwise arbitrary negative real number, and set*

$$c_1 := -\tfrac{1}{2} - \tfrac{\ln(1-a)}{\ln(b)} \quad \text{and} \quad c_2 := -\tfrac{1}{2\ln(b)} \,. \tag{40}$$

Let $\hat{\mathcal{D}}_3$ and $\hat{\mathcal{D}}'_3$ be the sets

$$\begin{aligned} \hat{\mathcal{D}}_3 &:= \{\, (u,v) \in \mathbb{R}^2 : u = c_2\, v^2 + c_1\, v + c_0 \,\}, \\ \hat{\mathcal{D}}'_3 &:= \hat{\mathcal{D}}_3 \cap \{(u,v) : u \le 0,\ v \ge 0\}. \end{aligned}$$

Then,

$$\hat{N}(\hat{\mathcal{D}}_3) \subset \hat{\mathcal{D}}_3 \quad \textit{and} \quad \hat{T}(\hat{\mathcal{D}}'_3) \subset \{\,(u,v) : u > c_2\, v^2 + c_1\, v + c_0\,,\ v > 0\}\,. \tag{41}$$

Figure 8 shows the curve $\hat{\mathcal{D}}'_3$ described in Lemma 3 along with its image under the map $\hat{T}$ and illustrates the relation $\hat{T}(\hat{\mathcal{D}}'_3) \subset \{\,(u,v) : u > c_2\, v^2 + c_1\, v + c_0\,,\ v > 0\}$.

***Proof*:** Let $(u,v) \in \hat{\mathcal{D}}_3$ (i.e. $u = c_2\, v^2 + c_1\, v + c_0$) and set

$$(u',v') := \hat{N}(u,v) = \big(\ln((1-a)\, e^{u+v}),\, \ln(\tfrac{1}{b}\, e^{v})\big)\,. \tag{42}$$

Then,

$$\begin{aligned} &c_2\,(v')^2 + c_1\, v' + c_0 \\ &\quad = c_2\,(\,v - \ln(b)\,)^2 + c_1\,(\,v - \ln(b)) + c_0 \\ &\quad = c_2\, v^2 - 2\,c_2\,(\ln(b))\, v + c_2\,(\ln(b))^2 + c_1\, v - c_1\, \ln(b) + c_0 \\ &\quad = u - 2\,c_2\,(\ln(b))\, v + c_2\,(\ln(b))^2 - c_1\, \ln(b). \end{aligned} \tag{43}$$

A straightforward calculation using (40) gives

$$-2\,c_2\,(\ln(b)) = 1 \quad \text{and} \quad c_2\,(\ln(b))^2 - c_1\, \ln(b) = \ln(1-a)\,. \tag{44}$$

Consequently, from (42), (43) and (44) we have

$$c_2\,(v')^2 + c_1\, v' + c_0 = u + v + \ln(1-a) = \ln(\,(1-a)\, e^{u+v}) = u'.$$

This proves the first relation in (41). To prove the second relation in (41), let $(u, v) \in \hat{\mathcal{D}}'_3$. Since $0 \le v \le q_2$ and $u \le 0$,

$$(1-b)e^{u+v} + b < \max\{e^{u+v}, 1\} \le \max\{e^{v}, 1\} = e^{v},$$

and it follows that

$$\ln\left(\frac{e^{v}}{(1-b)e^{u+v}+b}\right) > 0.$$

Consequently, $\hat{T}(u, v) \in \{(s, t) : t > 0\}$. Now, define

$$\mathcal{Q}_- := \{(u, v) \in \hat{\mathcal{D}}'_3 : u + v \le 0\} \quad \text{and} \quad \mathcal{Q}_+ := \{(u, v) \in \hat{\mathcal{D}}'_3 : u + v > 0\}. \tag{45}$$

Clearly, $\hat{\mathcal{D}}'_3 = \mathcal{Q}_- \cup \mathcal{Q}_+$. We consider two cases separately. If $(u, v) \in \mathcal{Q}_+$, then by Proposition 4 we have $(u, v) \preceq_{se} \hat{T}(u, v)$. Combining this with (38), it follows that $\hat{T}(u, v) \in \{(s, t) : s > c_2 t^2 + c_1 t + c_0\,,\ t > 0\}$. If now $(u, v) \in \mathcal{Q}_-$, then by Proposition 4, $(u, v) \preceq_{ne} \hat{T}(u, v)$. Also, note that for $x := e^u$ and $y := e^v$,

$$T(x, y) - N(x, y) = \left(a\,x\,, -\frac{(1-b)\,x\,y}{b(\,(1-b)\,x\,y + b)} \right).$$

Hence $N(x, y) \preceq_{se} T(x, y)$, which implies $\hat{N}(u, v) \preceq_{se} \hat{T}(u, v)$. Now, $\hat{N}(u, v) \in \hat{\mathcal{D}}_3$ by the first part of this proof and the relation $\hat{T}(u, v) \in \{(s, t) : s > c_2 t^2 + c_1 t + c_0\,,\ t > 0\}$ follows. The curve $\hat{\mathcal{D}}'_3$ along with its image under $\hat{T}$ can be seen in Figure 8. □

4.3. Construction of a family of compact sets

We begin by establishing some useful inequalities. We shall need the following values, which can be obtained from Equation (26):

$$\hat{f}_1(0) = \ln\left(\frac{(b-a)e^{\tau(r-1)}}{(b-a) + (1-b)(1-e^{\tau(r-1)})}\right) \tag{46}$$

and

$$\hat{f}_1'(0) = -\frac{1-b-(1-a)\,r\,e^{\tau\,(1-r)}}{1-b-(1-a)\,e^{\tau\,(1-r)}}. \tag{47}$$

Lemma 4, presented below, is easily established from relations (20), (25), (46) and (47).

Lemma 4: *There exists $\tau_1 > 0$ such that*

$$\hat{f}_1(0) < 0\,, \quad \textit{for} \quad \tau \ge \tau_1,$$

and

$$\hat{f}_1'(0) < 0\,, \quad \textit{for} \quad \tau \ge \tau_1.$$

The sets $\mathcal{K}_\tau$ are introduced next.

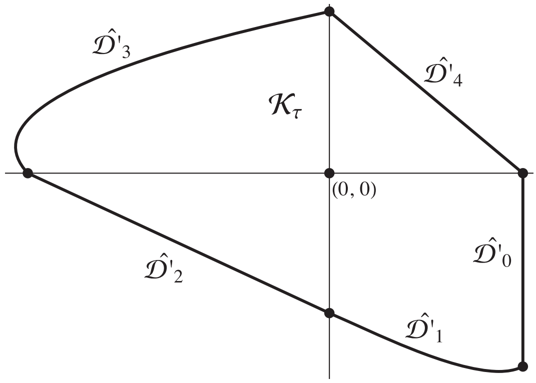

Figure 9. A set $\mathcal{K}_\tau$ whose boundary consists of the sets $\hat{\mathcal{D}}'_\ell$ for $0 \le \ell \le 4$.

Definition 1: Let $\tau \in \mathbb{R}_+$ be such that $\tau \ge \tau_1$ with τ_1 as in Lemma 4, and set

$$q_1 := \hat{f}_1(0), \tag{48}$$

$$p_2 := -\frac{\hat{f}_1(0)}{\hat{f}_1'(0)}, \quad \text{and}$$

$$q_2 := \frac{-c_1 + \sqrt{{c_1}^2 - 4\,c_2\,p_2}}{2\,c_2}, \tag{49}$$

where $\hat{f}_1(0), \hat{f}_1'(0)$, c_1 and c_2 are given in (40), (46), and (47). Let the set $\mathcal{K}_\tau$ be the convex hull of the sets $\hat{\mathcal{D}}'_\ell$, $0 \le \ell \le 4$, where

- $\hat{\mathcal{D}}'_0$ is the line segment joining $(\tau, 0)$ and $(\tau, -\tau)$.
- $\hat{\mathcal{D}}'_1$ is the curve given in Lemma 1 with endpoints $(\tau, -\tau)$ and $(0, q_1)$.
- $\hat{\mathcal{D}}'_2$ is the line segment with endpoints $(0, q_1)$ and $(p_2, 0)$.
- $\hat{\mathcal{D}}'_3$ is the parabolic arch in Lemma 3 with endpoints at $(p_2, 0)$ and $(0, q_2)$.
- $\hat{\mathcal{D}}'_4$ is the line segment with endpoints $(0, q_2)$ and $(\tau, 0)$.

Remark 1: In Definition 1, $q_1 < 0$ and $p_2 < 0$ by Lemma 4. Therefore, $\mathcal{K}_\tau$ is a compact and convex neighborhood of the origin such that $\partial\mathcal{K}_\tau = \cup_{\ell=0}^4 \hat{\mathcal{D}}'_\ell$. See Figure 9.

Remark 2: In order to simplify notation, dependence on τ has been suppressed in the terms q_1, p_1, q_2, and $\hat{\mathcal{D}}'_\ell$, $0 \le \ell \le 4$.

The proof of Proposition 3 involves an asymptotic argument on the parameter τ as it relates to the compact set $\mathcal{K}_\tau$. It is useful for us to first describe the asymptotic behaviour of q_1, q_2, and p_2 when $\tau \to +\infty$.

Claim 1: *The asymptotic behaviour of q_1, q_2, and p_2 is as follows:*

(i) $q_1 = (r-1)\tau + \mathcal{O}(1)$ *as* $\tau \to +\infty$.
(ii) $p_2 = \left(\frac{r-1}{r}\right)\tau + \mathcal{O}(1)$ *as* $\tau \to +\infty$.
(iii) $q_2 = \sqrt{2\,\ln(b)\,\frac{r-1}{r}}\,\sqrt{\tau} + \mathcal{O}(1)$ *as* $\tau \to +\infty$.

Proof: From (46) and (48),

$$\begin{aligned} q_1 &= (r-1)\tau + \ln(b-a) - \ln\left((b-a) + (1-b)(1-e^{\tau(r-1)})\right) \\ &= (r-1)\tau + \ln\left(\frac{1}{1+\frac{1-b}{b-a}\left(1-e^{\tau(r-1)}\right)}\right). \end{aligned}$$

Since $r \in (0,1)$, (i) follows. Similarly,

$$\begin{aligned} p_2 = -\frac{\hat{f}_1(0)}{\hat{f}_1'(0)} &= -\frac{1}{\hat{f}_1'(0)} \ln\left(\frac{(b-a)e^{\tau(r-1)}}{(b-a)+(1-b)(1-e^{\tau(r-1)})}\right) \\ &= \frac{(1-r)}{\hat{f}_1'(0)}\tau - \frac{\ln(b-a)}{\hat{f}_1'(0)} + \frac{\ln\left((b-a)+(1-b)(1-e^{\tau(r-1)})\right)}{\hat{f}_1'(0)}. \end{aligned}$$

Since $r \in (0,1)$ and $\lim_{\tau\to\infty} \hat{f}_1'(0) = -r$, (ii) follows. Finally, from (49) and (ii),

$$\begin{aligned} q_2 &= \frac{-c_1 + \sqrt{c_1^2 - 4\,c_2\,p_2}}{2c_2} \\ &= \sqrt{-\frac{p_2}{c_2}} + \mathcal{O}(1) \\ &= \sqrt{-\left(\frac{r-1}{c_2\,r}\right)\tau} + \mathcal{O}(1), \end{aligned}$$

and thus (iii) follows from substituting $c_2 = -1/(2\ln(b))$. □

4.4. *Proof of Proposition 3*

To prove Proposition 3, we establish first that any given bounded set $\mathcal{B} \subset \mathbb{R}^2$ is contained in $\mathcal{K}_\tau$ for τ large enough.

Claim 2: *Let $\mathcal{B} \subset \mathbb{R}^2$ be bounded. Then for all τ large enough, $\mathcal{B} \subset \mathcal{K}_\tau$.*

Proof: Since $\mathcal{K}_\tau$ is convex, the quadrilateral $\mathcal{S}$ whose endpoints are $(\tau, 0)$, $(0, q_1)$, $(p_2, 0)$ and $(0, q_2)$ is such that $\mathcal{S} \subset \mathcal{K}_\tau$ (see Figure 10). Therefore, Claim 1 implies that for all large enough τ, $\mathcal{K}_\tau$ contains $\mathcal{B}$. □

Next we prove that for all τ large enough, $\hat{T}(\hat{\mathcal{D}}'_\ell) \subset \mathcal{K}_\tau$ for $0 \le \ell \le 4$. Once this has been established, it follows that $\mathcal{K}_\tau$ is $\hat{T}$-invariant for large τ and the proof of Proposition 3 will be complete. The boundary of $\mathcal{K}_\tau$ along with its image under the map $\hat{T}$ can be seen in Figure 11. We assume in Claims 3 through 7 that $\tau \ge \tau_1$.

Claim 3: $\hat{T}(\hat{\mathcal{D}}'_0) \subset \mathcal{K}_\tau$.

Proof: Let us first verify that the endpoints of $\hat{T}(\hat{\mathcal{D}}'_0)$, namely the points $\hat{T}(\tau, 0)$ and $\hat{T}(\tau, -\tau)$, belong to $\mathcal{K}_\tau$. Notice $\hat{T}(\tau, 0) = \left(\tau, -\ln\left((1-b)\,e^\tau + b\right)\right)$ satisfies $-\ln\left((1-b)\,e^\tau + b\right) > -\tau$, hence $\hat{T}(\tau, 0) \in \mathcal{K}_\tau$. Also, $\hat{T}(\tau, -\tau) = (\ln(a\,e^\tau + (1-a)), -\tau)$ satisfies $0 < \ln(a\,e^\tau + (1-a)) < \tau$. Since $(\tau, -\tau) \in \hat{\mathcal{D}}'_1$, it follows from Lemma 1 that $\hat{T}(\tau, -\tau) \in \mathcal{K}_\tau$, so both endpoints of $\hat{T}(\hat{\mathcal{D}}'_0)$ belong to $\mathcal{K}_\tau$.

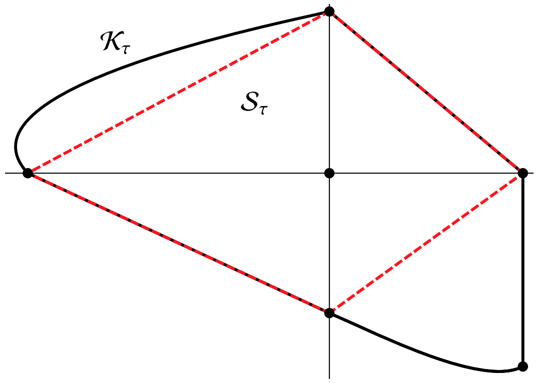

Figure 10. The quadrilateral $\mathcal{S}_\tau$ with $\mathcal{S}_\tau \subset \mathcal{K}_\tau$.

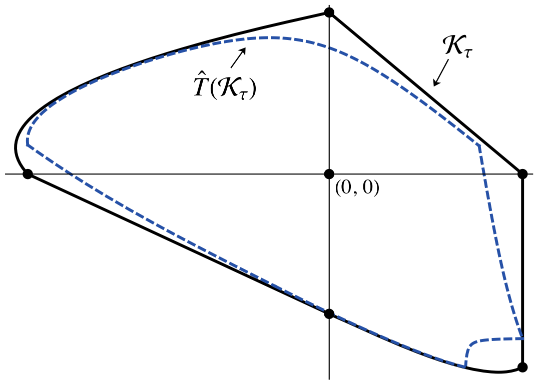

Figure 11. Boundary of the set $\mathcal{K}_\tau$ (solid) and its image under $\hat{T}$ (dashed).

We now show that $\hat{T}(\hat{\mathcal{D}}_0')$ is a curve linearly ordered in the $\preceq_{ne}$ partial order. We may write $\hat{\mathcal{D}}_0' = \{(\tau, -\tau\, t) : \ 0 \le t \le 1\}$. For $0 \le t \le 1$ set

$$(\tilde{u}(t), \tilde{v}(t)) := \hat{T}((\tau, -\tau\, t)) = \left(\ln\left(a\, e^{\tau} + (1-a)\, e^{\tau(1-t)}\right), \ln\left(\frac{e^{-\tau t}}{(1-b)\, e^{\tau(1-t)} + b}\right)\right). \tag{50}$$

Then $\hat{T}(\hat{\mathcal{D}}_0') = \{(\tilde{u}(t), \tilde{v}(t)) : \ 0 \le t \le 1\}$. From (50),

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} = -\frac{(1-a)\tau\, e^{\tau(1-t)}}{(1-a)\, e^{\tau(1-t)} + a\, e^{\tau}} < 0 \quad \text{and} \quad \frac{\mathrm{d}\tilde{v}}{\mathrm{d}t} = -\frac{b\, \tau\, e^{\tau t}}{(1-b)e^{\tau} + b\, e^{\tau t}} < 0.$$

Thus both $\tilde{u}(t)$ and $\tilde{v}(t)$ are decreasing functions of t in $[0, 1]$, so $\hat{T}(\hat{\mathcal{D}}_0')$ is a curve linearly ordered in the $\preceq_{ne}$ partial order. It follows that $\hat{T}(\hat{\mathcal{D}}_0')$ is a subset of the closed rectangular region $\mathcal{R}$ determined by the vertices $\hat{T}(\tau, 0)$ and $\hat{T}(\tau, -\tau)$. Since the second coordinate of $\hat{T}(\tau, -\tau)$ is equal to $-\tau$ and $\hat{\mathcal{D}}_1'$ is the graph of a convex function, it follows from (27) that $\mathcal{R} \subset \mathcal{K}_\tau$, and consequently, $\hat{T}(\hat{\mathcal{D}}_0') \subset \mathcal{K}_\tau$. □

Claim 4: $\hat{T}(\hat{\mathcal{D}}_1') \subset \mathcal{K}_\tau$.

***Proof*:** For $(u,v) \in \hat{\mathcal{D}}_1'$ arbitrary but fixed, let $(\tilde{u},\tilde{v})$ be given by

$$(\tilde{u},\tilde{v}) = \hat{T}(u,v) = \left(\ln\left(a\,e^{u} + (1-a)\,e^{u+v}\right),\ \ln\left(\frac{e^{v}}{(1-b)\,e^{u+v}+b}\right)\right).$$

By the second relation in (27) of Lemma 1, and convexity of $\mathcal{K}_\tau$ and $\hat{\mathcal{D}}_1$, it is sufficient to verify that $\tilde{v} < 0$. Notice $(u,v) \in \hat{\mathcal{D}}_1'$ implies $u > 0$, $v < 0$, and $(1-b)\,e^{u+v} + b > (1-b)\,e^{v} + b > e^{v}$. It follows that $\tilde{v} < 0$. □

Claim 5: $\hat{T}(\hat{\mathcal{D}}_2') \subset \mathcal{K}_\tau$.

***Proof*:** The line segment $\hat{\mathcal{D}}_2'$ has slope $-\frac{q_1}{p_2} = \hat{f}_1'(0)$. Now

$$\hat{f}_1'(0) + r = \frac{(1-b)(1-r)\,e^{r\tau}}{(b-1)e^{r\tau} + (1-a)\,e^{\tau}} = \frac{(1-b)(1-r)}{(b-1) + (1-a)\,e^{\tau(1-r)}} > 0.$$

Thus the hypothesis $-\frac{q_1}{p_2} > -r$ of Lemma 2 is satisfied. Now, let $\mathcal{L}$ be the line through $(0,q_1)$ and $(p_2,0)$ and let $\mathcal{L}_0$ be the connected component of $\mathbb{R}^2 \setminus \mathcal{L}$ that contains the origin. Lemma 2 guarantees $\hat{T}(\hat{\mathcal{D}}_2') \subset \mathcal{L}_0$. Also, note $\hat{T}(\hat{\mathcal{D}}_2')$ is linearly ordered in the $\preceq_{se}$ partial order, which we verify next. We may write $\hat{\mathcal{D}}_2' = \{(p_2\,t, (1-t)\,q_1) : 0 \le t \le 1\}$. For $0 \le t \le 1$ set

$$\begin{aligned}(\tilde{u}(t),\tilde{v}(t)) &:= \hat{T}((p_2\,t,(1-t)\,q_1))\\ &= \left(\ln\left(a\,e^{p_2\,t} + (1-a)\,e^{p_2\,t+(1-t)\,q_1}\right),\ \ln\left(\frac{e^{(1-t)\,q_1}}{(1-b)\,e^{p_2\,t+(1-t)\,q_1}+b}\right)\right),\end{aligned} \tag{51}$$

then $\hat{T}(\hat{\mathcal{D}}_2') = \{(\tilde{u}(t),\tilde{v}(t)) : 0 \le t \le 1\}$. From (51),

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} = \frac{a\,p_2 + (1-a)(p_2-q_1)e^{(1-t)\,q_1}}{a + (1-a)\,e^{(1-t)\,q_1}} \quad \text{and} \quad \frac{\mathrm{d}\tilde{v}}{\mathrm{d}t} = -\frac{p_2\,(1-b)e^{q_1+p_2\,t} + b\,q_1\,e^{q_1\,t}}{(1-b)e^{q_1+p_2\,t} + b\,e^{q_1\,t}}. \tag{52}$$

Using statements (i) and (ii) of Claim 1 and (52) we conclude that for τ large enough, $\tilde{u}(t)$ is a decreasing function of $t \in [0,1]$ and $\tilde{v}(t)$ is an increasing function of $t \in [0,1]$. Consequently, $\hat{T}(\hat{\mathcal{D}}_2')$ is a curve linearly ordered in the $\preceq_{se}$ partial order and is thus a subset of the rectangular region $\mathcal{R}$ determined by the initial and final points. Hence

$$\hat{T}(\hat{\mathcal{D}}_2') \subset \mathcal{R} \cap \mathcal{L}_0. \tag{53}$$

Note that $\hat{T}(0,q_1) \in \mathcal{K}_\tau$ by Claim 4 and $\hat{T}(p_2,0) \in \mathcal{K}_\tau$ by Lemma 3. It follows from (53) and the convexity of $\mathcal{K}_\tau$ that $\hat{T}(\hat{\mathcal{D}}_2') \subset \mathcal{K}_\tau$. □

Claim 6: *For all τ large enough, $\hat{T}(\hat{\mathcal{D}}_3') \subset \mathcal{K}_\tau$.*

***Proof*:** Suppose $(u, v) \in \hat{\mathcal{D}}'_3$. From (22),

$$\hat{T}(u,v) = \left(\ln\left(ae^u + (1-a)e^{u+v}\right), \ln\left(\frac{e^v}{(1-b)e^{u+v}+b}\right)\right).$$

By statement (41) in Lemma 3,

$$\hat{T}(u,v) \in \{(s,t) : s > c_2\, t^2 + c_1\, t + c_0\,,\ \ t > 0\}. \tag{54}$$

Now, let $\mathcal{L}$ be the line through $(0, q_2)$ and $(\tau, 0)$. Then $\mathbb{R}^2 \setminus \mathcal{L}$ has two connected components, one of which, $\mathcal{L}_0$, contains the origin. As a result of (54), to complete the proof it suffices to verify that $\hat{T}(u, v)$ belongs to $\mathcal{L}_0$. In other words, for τ large enough,

$$\frac{1}{\tau}\ln\left(ae^u + (1-a)e^{u+v}\right) + \frac{1}{q_2}\ln\left(\frac{e^v}{(1-b)e^{u+v}+b}\right) < 1. \tag{55}$$

Set

$$\Delta_\tau := 1 - \frac{1}{\tau}\ln\left(ae^u + (1-a)e^{u+v}\right) - \frac{1}{q_2}\ln\left(\frac{e^v}{(1-b)e^{u+v}+b}\right). \tag{56}$$

Then (55) is equivalent to

$$\Delta_\tau > 0, \tag{57}$$

for τ large enough. Consider the sets $\mathcal{Q}_+$ and $\mathcal{Q}_-$ defined in (45). We verify (57) for $(u, v) \in \mathcal{Q}_-$ and for $(u, v) \in \mathcal{Q}_+$ separately. Suppose $(u, v) \in \mathcal{Q}_-$. In this case, $a\,e^u + (1-a)e^{u+v}$ is a weighted average of two numbers that are less than 1 which implies that $\ln\left(ae^u + (1-a)e^{u+v}\right) < 0$. Consequently, $\hat{T}(u, v) \in \{(s, t) : s \le 0, t > 0\}$. Combining this with (54), (57) follows. Now suppose $(u, v) \in \mathcal{Q}_+$. From (56),

$$\begin{aligned}\tau\, q_2\, \Delta_\tau &= \tau q_2 - q_2 \ln\left(a\,e^u + (1-a)e^{u+v}\right) - \tau \ln\left(\frac{e^v}{(1-b)e^{u+v}+b}\right)\\ &= \tau(q_2 - v) + \tau\ln\left((1-b)e^{u+v} + b\right) - q_2(u+v) - q_2\ln\left(ae^{-v} + 1 - a\right).\end{aligned} \tag{58}$$

By convexity, $e^{(1-b)(u+v)+b\cdot 0} \le (1-b)e^{u+v} + b\,e^0$. That is,

$$(1-b)(u+v) \le \ln\left((1-b)e^{u+v} + b\right). \tag{59}$$

Recognizing that $\ln\left(ae^{-v} + 1 - a\right) < 0$ for $v \ge 0$ and combining (58) and (59),

$$\begin{aligned}\tau\, q_2\, \Delta &\ge \tau(q_2 - v) + \tau(1-b)(u+v) - q_2(u+v)\\ &= \tau(q_2 - v) + (u+v)(\tau(1-b) - q_2).\end{aligned} \tag{60}$$

From Claim 1, we can consider τ large enough such that $q_2 < (1-b)\tau$. Therefore, since $u + v > 0$ and $0 < v \le q_2$, (60) implies $\Delta_\tau \ge 0$. If $\Delta_\tau = 0$, then (60) implies $v = q_2$ and $u + v = 0$, which contradicts $(u, v) \in \mathcal{Q}_+$. Consequently, (57) holds. □

Claim 7: *For all τ large enough, $\hat{T}(\hat{\mathcal{D}}'_4) \subset \mathcal{K}_\tau$.*

Proof: We have $\hat{\mathcal{D}}'_4 = \{(t\,\tau, (1-t)\,q_2) : t \in [0,1]\}$. For $t \in [0,1]$, let $(\tilde{u}(t), \tilde{v}(t))$ be given by

$$(\tilde{u}(t), \tilde{v}(t)) = \hat{T}(t\,\tau, (1-t)\,q_2) \tag{61}$$
$$= \left(\ln\left(ae^{t\tau} + (1-a)e^{t\tau+(1-t)q_2}\right), \ln\left(\frac{e^{(1-t)q_2}}{(1-b)e^{t\tau+(1-t)q_2} + b}\right)\right).$$

From (61),

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} = \frac{(1-a)(\tau - q_2)e^{q_2} + a\,\tau\,e^{q_2 t}}{(1-a)e^{q_2} + a\,e^{q_2 t}}, \tag{62}$$
$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t} = -\frac{\tau\,e^{q_2+t\tau}(1-b) + b\,q_2\,e^{q_2 t}}{b\,e^{q_2 t} + (1-b)e^{q_2+t\tau}}. \tag{63}$$

Using statement (iii) of Claim 1 along with (62) and (63), we can conclude for τ large enough that $\tilde{u}(t)$ is an increasing function of $t \in [0,1]$ and $\tilde{v}(t)$ is a decreasing function of $t \in [0,1]$. As a consequence,

$$\hat{T}(\hat{\mathcal{D}}'_4) \text{ is linearly ordered in the } \preceq_{se} \text{ partial order.} \tag{64}$$

We also have

$$\hat{T}(\tau, 0) = \left(\ln\left(ae^{\tau} + (1-a)e^{\tau}\right), \ln\left(\frac{1}{(1-b)e^{\tau} + b}\right)\right)$$
$$= \left(\tau, -\ln\left((1-b)e^{\tau} + b\right)\right) \in \hat{\mathcal{D}}'_0. \tag{65}$$

In light of (64) and (65), to prove the claim, it is sufficient to verify that $\hat{T}(\hat{\mathcal{D}}'_4)$ is in a suitable component of the complement of the line through $(0, q_2)$ and $(\tau, 0)$, for τ large enough. More precisely, we wish to verify

$$\frac{1}{\tau}\ln\left(ae^{t\tau} + (1-a)e^{t\tau+(1-t)q_2}\right) + \frac{1}{q_2}\ln\left(\frac{e^{(1-t)q_2}}{(1-b)e^{t\tau+(1-t)q_2} + b}\right) < 1. \tag{66}$$

For fixed τ, define

$$\psi_\tau(t) := q_2 \ln\left(a + (1-a)e^{(1-t)q_2}\right) - \tau\ln\left((1-b)e^{t\tau+(1-t)q_2} + b\right).$$

Equation (66) is equivalent to

$$\psi_\tau(t) < 0 \quad \text{for} \quad 0 \le t \le 1. \tag{67}$$

We have,

$$\psi'_\tau(t) = -e^{(1-t)q_2}\left(\frac{(1-b)\tau(\tau - q_2)e^{t\tau}}{b + (1-b)e^{t\tau+(1-t)q_2}} + \frac{(1-a)q_2^2}{a + (1-a)e^{(1-t)q_2}}\right) < 0. \tag{68}$$

Finally, $\psi_\tau(0) < 0$ follows from Claim 6 and thus (68) implies (67). □

This completes the proof of Proposition 3 and thus, by Corollary 2, of statement (vii) from Theorem 1.

Note

1. We do not know any easily testable conditions on α, γ with $1 < \gamma < \alpha$ to determine whether nontrivial periodic solutions exist.

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