CH 9.3 SEPARATION of VARIABLES

GOAL: Now that we are familiar with the terminology of DIFFERENTIAL EQUATIONS, we can learn one the main techniques that can be used to actually solve a differential equation: SEPARATION OF VARIABLES.

PART 1: WHAT IS A SEPARABLE DIFFERENTIAL EQ?

* Our focus in this section will be on first-order differential equations (I.e. DEs with only a first derivative) that have an important property:

DEFN: [SEPARABLE DE]:

A DIFF EQ THAT CAN BE WRITTEN AS

$$\frac{dy}{dx} = f(x) \cdot g(y) \quad \text{OR} \quad \frac{dy}{dx} = \frac{f(x)}{g(y)} \quad g(y) \neq 0$$

Ex. Which of the following differential equations are SEPARABLE?

[A] $dy/dx = x^2y + y = y(x^2+1)$ ✓

[B] $dy/dx = x + y$ ✗

[C] $\frac{dy}{dx} = e^{x^2+y} = e^{x^2}e^{y}$ ✓

[D] $dy/dx = \sqrt{y^5+8}$ ✓

[E] $dy/dx = x^2y - x$ ✗

[F] $dy/dx = 2x^3$ ✓

PART 2: THE METHOD of SEPARATION of VARIABLES

! APPLIES ONLY TO SEPARABLE D.E.S

STEP 1: WRITE ALL y's ON LEFT & x's ON RIGHT (ALGEBRA)
dy/dx CAN BE TREATED AS A FRACTION!

STEP 2: INTEGRATE BOTH SIDES.

STEP 3: SOLVE FOR y (IF POSSIBLE).

Ex 2. Find the <u>general solution</u> to the following differential equations using SEPARATION OF VARIABLES

General solution is also known as <u>family of solutions</u>

A $\frac{dy}{dx} = \frac{x^2}{y^2}$

Sol:

(1) $y^2\,dy = x^2\,dx$

(2) $\int y^2\,dy = \int x^2\,dx$

$\frac{y^3}{3} + C_1 = \frac{x^3}{3} + C_2$ ARBITRARY

(3) $\frac{y^3}{3} = \frac{x^3}{3} + C_2 - C_1$

$\frac{y^3}{3} = \frac{x^3}{3} + C$

$y^3 = x^3 + 3C$ $=C$

$$\boxed{y = \sqrt[3]{x^3 + C}}$$

CHECK!

LHS: $\frac{dy}{dx} = \frac{1}{\cancel{3}}(x^3+C)^{-2/3} \cdot \cancel{3}x^2 = \frac{x^2}{(x^3+C)^{2/3}}$

RHS: $\frac{x^2}{y^2} = \frac{x^2}{(\sqrt[3]{x^3+C})^2}$ ✓

B $(y+xy)y' = 2$

Sol:

$(y+xy)\frac{dy}{dx} = 2 \Rightarrow y(1+x)\frac{dy}{dx} = 2$

$\int y\,dy = \int \frac{2}{1+x}\,dx$ $x \neq -1$

$\frac{y^2}{2} = 2\ln|1+x| + C$

$$\boxed{y^2 = 4\ln|1+x| + C}$$ IMPLICIT GENERAL SOLN.

$$\boxed{y = \pm\sqrt{4\ln|1+x| + C}}$$ EXPLICIT GENERAL SOLN.

C $\frac{dy}{dx} = \frac{2x^2}{y+\sin(y)}$ $y \neq 0.$

sol:

$$\int (y+\sin(y))\,dy = \int 2x^2\,dx$$

$$\boxed{\frac{y^2}{2} - \cos(y) = \frac{2x^3}{3} + C}$$

D $\frac{dy}{dt} = t^2 y$ $\xrightarrow{y \neq 0}$ $\int \frac{dy}{y} = \int t^2\,dt$

sol: if $y=0$

$\frac{dy}{dt} = 0$ ✓

so $y=0$ IS EQUILIBRIUM

$$e^{\ln|y|} = e^{\left(\frac{t^3}{3} + C\right)}$$

$$|y| = e^C e^{\frac{t^3}{3}}$$

$$y = \pm e^C e^{\frac{t^3}{3}}$$

$$\boxed{y = Ce^{t^3/3}}$$

E $xy'+4=y^2 \longrightarrow x\frac{dy}{dx}=y^2-4 \longrightarrow \frac{dy}{y^2-4}=\frac{dx}{x}$

sol:

$$\int\frac{dy}{y^2-4}=\int\frac{dx}{x} \Rightarrow \int\frac{dy}{(y+2)(y-2)}=\int\frac{dx}{x}$$

$$\int\frac{-1/4}{y+2}+\frac{1/4}{y-2}\,dy=\int\frac{dx}{x} \Rightarrow -\frac{1}{4}\ln|y+2|+\frac{1}{4}\ln|y-2|=\ln|x|+C$$

$$\frac{1}{4}(\ln|y-2|-\ln|y+2|)=\ln|x|+C$$

$$\frac{1}{4}\left(\ln\left|\frac{y-2}{y+2}\right|\right)=\ln|x|+C$$

$$e^{\ln\left|\frac{y-2}{y+2}\right|}=e^{(4\ln|x|+C)}$$

$$\left|\frac{y-2}{y+2}\right|=e^Ce^{4\ln|x|}$$

$$\frac{y-2}{y+2}=\pm e^Ce^{\ln(x^4)}$$

$$\frac{y-2}{y+2}=\frac{Cx^4}{1}$$

$$\frac{y-2}{y+2-(y-2)}=\frac{Cx^4}{1-Cx^4}$$

$$\frac{y-2}{4}=\frac{Cx^4}{1-Cx^4}$$

$$\boxed{y=\frac{4Cx^4}{1-Cx^4}+2}$$

! $\frac{a}{b}=\frac{c}{d}$, $\frac{a}{b+a}=\frac{c}{d+c}$

$\frac{2}{3}=\frac{4}{6}$

$\frac{2}{5}=\frac{4}{10}$

Ex 3: Find the particular solution of each differential equation that satisfies the given initial condition (i.e. solve each **INITIAL VALUE PROBLEM**)

A $dy/dx=\frac{x^2}{y^2}$, $y(0)=4$

sol: GEN SOLN (from ABOVE) $\boxed{y=\sqrt[3]{x^3+C}}$

APPLY INITIAL COND $y(0)=4$: $4=\sqrt[3]{C}$ SO $C=64$

PARTICULAR SOLN $\boxed{y=\sqrt[3]{x^3+64}}$

B $(y+xy)y'=2$, $y(0)=3$

sol: GEN SOLN (from ABOVE). $\boxed{y=\pm\sqrt{4\ln|1+x|+C}}$

APPLY INITIAL COND $y(0)=3$: $3=\sqrt{4\ln|1|+C}$, $\ln|1|=0$

$C=9$

PART. SOLN. $\boxed{y=\sqrt{4\ln|1+x|+9}}$

PART 3: ORTHOGONAL TRAJECTORIES
(AKA PERPENDICULAR)

** Given a family of solution curves that satisfy a differential equation. We will find another collection of curves that are perpendicular to the original family. These new curves are called ORTHOGONAL TRAJECTORIES.

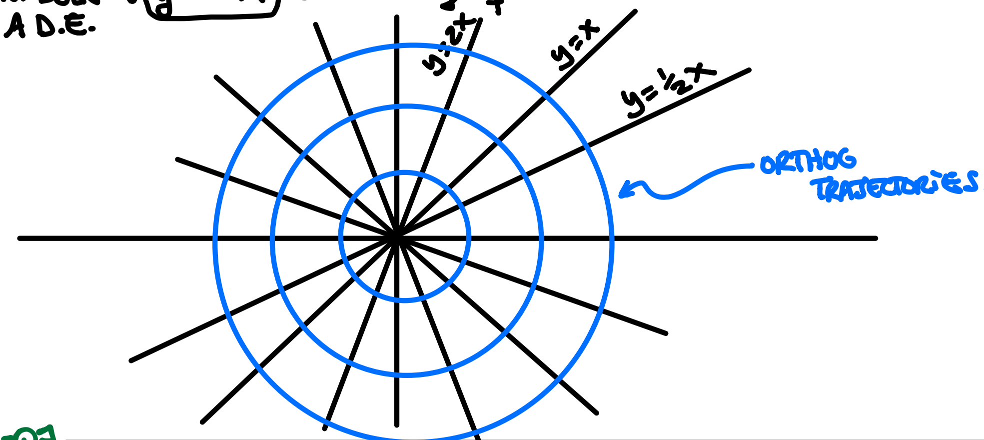
DEFN: [ORTHOGONAL TRAJECTORY]

An orthogonal trajectory of a family of curves is a curve that intersects each member of original family perpendicularly.

Some APPLICATIONS

In electric fields, force lines are $\perp$ to potential curves.

Ex. Gen. soln $\Rightarrow$ $\boxed{y = Cx}$ — family of curves. of a D.E.



PROCEDURE TO FIND ORTHOGONAL TRAJECTORIES

Given a family of curves.

(1) find a D.E. the family satisfies $\frac{dy}{dx} = \square$

(2) Eliminate the arbitrary constant by using original eq.

(3) find ORTHOGONAL D.E. by taking negative reciprocal

(4) Solve it to find orthog. trajectories.

Ex 4. Find the ORTHOGONAL TRAJECTORIES of the family of curves given by:

A $y = kx^3$ ← Solve for k: $\boxed{k = \frac{y}{x^3}}$

Sol: (1) $\frac{dy}{dx} = 3kx^2$

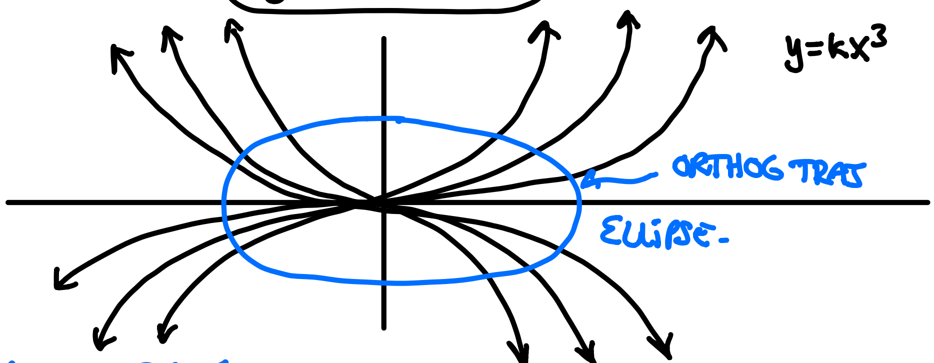
(2) $\frac{dy}{dx} = 3 \cdot \frac{y}{x^3} \cdot x^2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{3y}{x}}$ D.E of orig family.

(3) $\boxed{\frac{dy}{dx} = \frac{-x}{3y}}$ D.E of orthog. trajectories.

(4) Solve it: $\int y\,dy = \int \frac{-x}{3}\,dx$

$$\frac{y^2}{2} = \frac{-x^2}{6} + C$$

$$\boxed{3y^2 + x^2 = C}$$ orthog trajs.



Part 4: APPLICATIONS

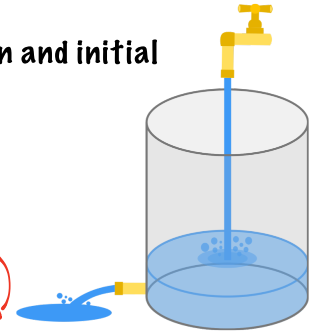
* Let's look at a few more applications of differential equations. We will determine appropriate differential equations to model certain situations and we will then be able to solve them!

NOTE: y = some quantity.

$$\frac{dy}{dt} = \text{RATE IN} - \text{RATE OUT}$$

Change in y over t → $\frac{dy}{dt}$

Ex 5. A tank contains 30 lbs of sugar dissolved in 200 gallons of water. A solution that contains 0.04 lbs of sugar per gallon of water is pumped into the tank at a rate of 20 gallons/min. The solution is the tank is kept thoroughly mixed, and is drained at the same rate (20 gallons/min). Let y(t) represent the amount of sugar (in lbs) in the tank after t minutes.



[A] Write an IVP that models this situation (i.e. a differential equation and initial condition)

Sol

$$dy/dt = \underset{(\text{lbs/min})}{\text{RATE of SUGAR COMING IN}} - \underset{(\text{lbs/min})}{\text{RATE of SUGAR GOING OUT}}$$

$$= \left(\text{CONCENTRATION COMING IN } (\text{lbs/gal})\right)\cdot\left(\text{RATE of FLOW IN } (\text{gal/min})\right) - \left(\text{CONCENTRATION GOING OUT } (\text{lbs/gal})\right)\cdot\left(\text{RATE of FLOW OUT } (\text{gal/min})\right)$$

$$= (0.04)(20) - \left(\frac{y}{200}\right)(20)$$

IVP: $\boxed{dy/dt = 0.8 - \frac{y}{10}}$ $\boxed{y(0) = 30}$

[B] Solve the DE from part [A]

Sol:

$$\frac{dy}{dt} = \frac{8-y}{10}$$ SOLVE it!

$$\boxed{y = 8 - Ce^{-t/10}}$$

find C

$y(0)=30$

$$30 = 8 - Ce^{0} \Rightarrow C = -22$$

$$\boxed{y = 8 + 22e^{-t/10}}$$ PART SOLN.

[C] How much sugar is in the tank after 20 minutes?

Sol:

$$\boxed{\begin{aligned} y(20) &= 8 + 22e^{-2} \text{ lbs} \\ &= 10.98 \text{ lbs.} \end{aligned}}$$