# CH 9.1: MODELING w/ DIFFERENTIAL EQ's

**GOAL:** We will learn what a DIFFERENTIAL EQUATION (DE) is and will explore some basic properties of them. We will also learn how DEs can be used to model certain real world situations.

## PART 1: BASICS of DEs

**DEFN: [DIFFERENTIAL EQ]:**

A DIFFERENTIAL EQ IN VARIABLE "$y$" (DEPENDS ON "$x$") IS AN EQUATION THAT INVOLVE SOME OF THE FOLLOWING

- $y$
- * DERIVATIVES of $y$ $(y', y'', y''', \dots)$ — at least 1
- $x$ or fns of $x$.

**Ex 1:** The following are differential equations

[A] $y'' + y' = e^x$ ORDER 2

[B] $y^{(13)} = x^2$ ORDER 13

**NOTE:** The ORDER of a Differential Equation is the power of the highest derivative.

**Q:** What does it mean to find a SOLUTION of a differential equation?

**A:** A SOLUTION of a DE is a FUNCTION $y(x)$ such that both sides of the DE are equal.

**Ex 2.** Verify that each function is a solution to the specified DIFFERENTIAL EQUATION.

[A] $y = 12x^4$. DE: $xy' = 4y$ (LHS: $xy'$, RHS: $4y$) ?

Sol:

$y = 12x^4$
$y' = 48x^3$

LHS: *CONVERT TO $x$*

$xy' = x \cdot 48x^3$
$= 48x^4$

RHS:

$4y = 4 \cdot (12x^4)$
$= 48x^4$

✓

So $y = 12x^4$ IS A SOLN of THE D.E.

[B] $y = Ae^{\lambda x}$ DE: $y' = \lambda y$ ?

Sol:

$y = Ae^{\lambda x}$
$y' = A\lambda e^{\lambda x}$

LHS: $A\lambda e^{\lambda x}$

RHS: $\lambda \cdot Ae^{\lambda x}$

✓

YES $y = Ae^{\lambda x}$ IS A SOLN.

$\boxed{C}$ $y = Ce^{3t} - \frac{1}{3}$, D.E: $\boxed{\frac{dy}{dt} = 1 + 3y}$ $y'$

Sol:

There are two different terms that are used for solutions to differential equations: PARTICULAR SOLUTIONS and the GENERAL SOLUTION

ONE

A PARTICULAR SOL'N IS A SOLUTION THAT DOES NOT INVOLVE AN ARBITRARY CONSTANT "C"

ALL

A GENERAL SOLUTION TO A D.E. IS A family of Solutions THAT INVOLVES AN ARBITRARY CONSTANT "C"

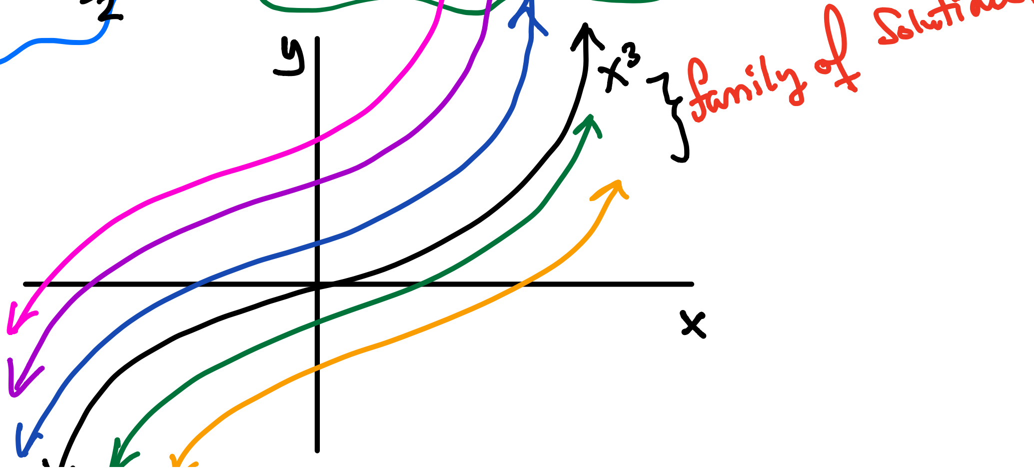
Ex. For the differential equation: $y' = 3x^2$
Find 3 PARTICULAR SOLUTIONS and then find the GENERAL SOLUTION

Sol: PART. SOL'NS:

$y = x^3$
$y = x^3 + 3$
$y = x^3 + \frac{1}{2}$

GEN SOL'N:

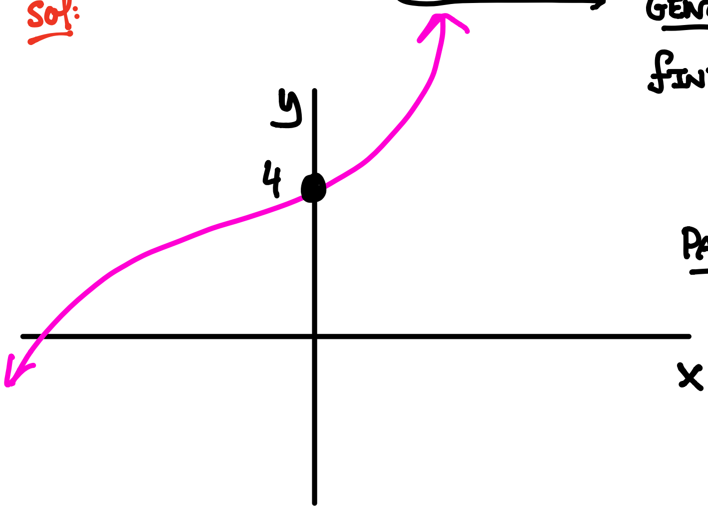
$y = x^3 + C.$



* A common type of question in the field of differential equations is called an **INITIAL VALUE PROBLEM (IVP).** Here, you will be given a differential equation and an initial condition that the function (i.e. solution) needs to satisfy. The goal is to find a **PARTICULAR SOLUTION** that satisfies the differential equation AND the initial condition.

**Ex 4.** Solve the **IVP**: $y' = 3x^2, \; y(0) = 4.$

**Sol:**




GENERAL SOLN $y = x^3 + C$

FIND "C" USING $y(0) = 4$

$4 = 0^3 + C \Rightarrow C = 4$

PARTICULAR SOLN

$$\boxed{y = x^3 + 4}$$

**Ex 5:** The general solution to a differential equation is given by: $y = Ce^{4x}$

Find the particular solution that satisfies the initial condition: $\underbrace{y(0) = 5}_{\text{INITIAL CONDITION}}$

**Sol:**

$y = Ce^{4x}$

$5 = Ce^0$

$C = 5$

PARTICULAR SOLN.

$$\boxed{y = 5e^{4x}}$$

**NOTE:** If a particular solution has the form y=K for some fixed constant K then we call this solution a CONSTANT SOLUTION (aka equilibrium solution). To find all constant solutions:

1. Let y=K for a fixed constant K
2. Input y=K into the DE (Note y'=0)
3. Solve for K (if possible)

**Ex 6** Find all CONSTANT SOLUTIONS of each differential equation (if any exist)

**A** $y' = y(y-2)$

sol: $0 = k(k-2)$

Solve for k $k=0,\ k=2$

$y=0$ $y=2$

EQUILIBRIA.

**B** $\frac{dy}{dt} = y^4 - 8y^3 + 12y^2$ $y=k$

sol: $0 = k^4 - 8k^3 + 12k^2$

Solve for k. $0 = k^2(k^2 - 8k + 12)$

$k=0,6,2$ $0 = k^2(k-6)(k-2)$

$y=0$ $y=6$ $y=2$

EQUILIBRIA.

! NOT ALL D.E's HAVE EQUILIBRIUM SOLNS.

**C** $dy/dx = xy^2$ $y=k$

sol: $0 = x k^2$

VARIABLE. (CAN BE ANYTHING)

$k=0$

$y=0$

**D** $dP/dt = 1 + P\cdot t$ $P=k$

sol: $0 = 1 + k\cdot t$

$\cancel{k = -1/t}$ NOT CONSTANT

NO EQUIL SOLNS

# PART 2: BEHAVIOR of SOLUTIONS

** Just by glancing at a differential equation, we can tell a lot about the BEHAVIOR of SOLUTIONS of the DE.

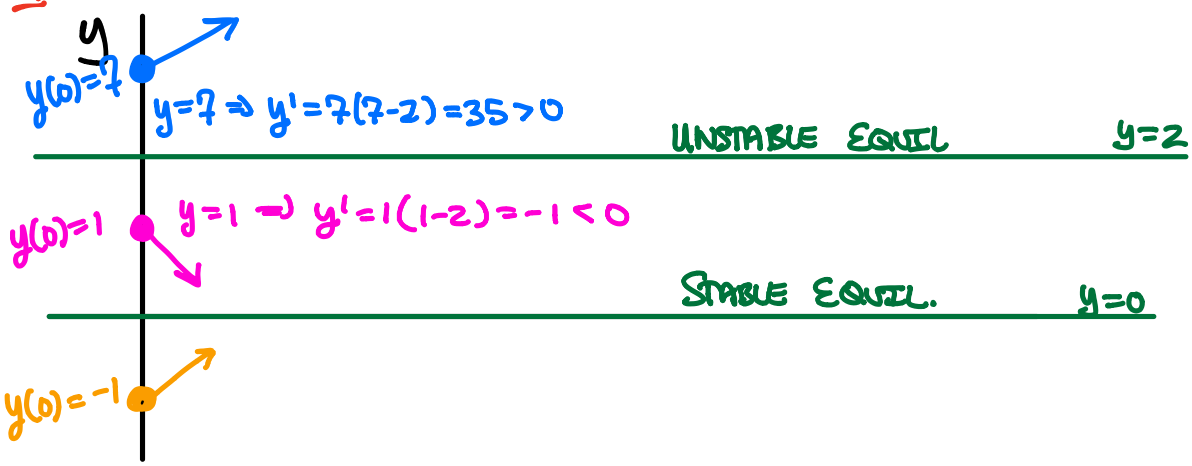
$$\frac{dy}{dx} > 0 \implies y \text{ IS INCREASING!}$$

$$\frac{dy}{dx} < 0 \implies y \text{ IS DECREASING!}$$

**Ex 7.** For each differential equation, sketch a graph that depicts the behavior of solutions to the DE.

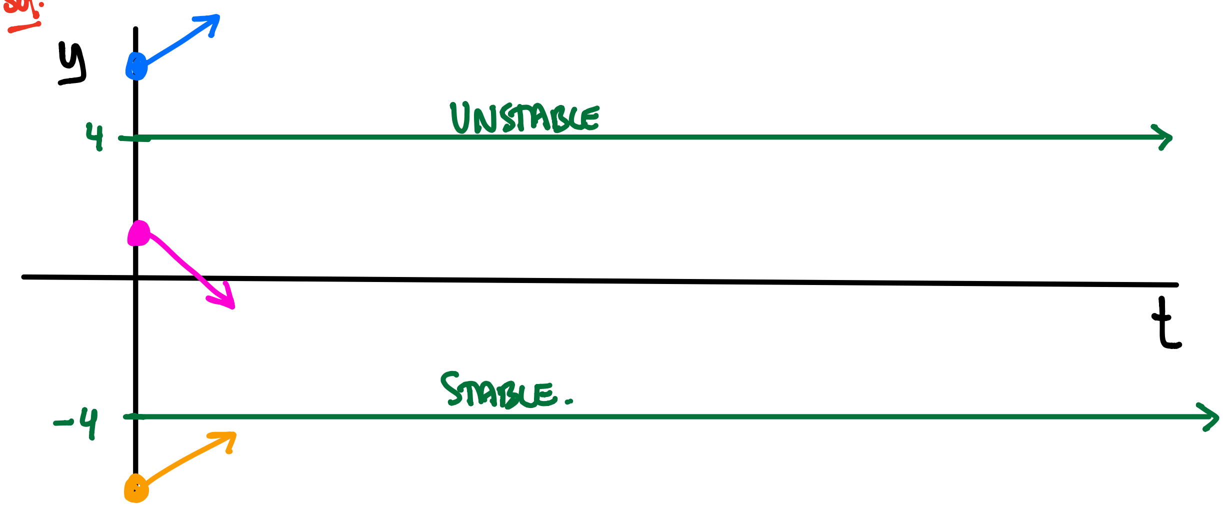
$\boxed{A}$ $y' = y(y-2)$ EQUIL SOLNS $y=0$ $y=2$

Sol:




$\boxed{B}$ $\frac{dy}{dt} = y^2 - 16 = (y+4)(y-4)$ $y=-4$ $y=4$

Sol:



PART 3: APPLICATIONS

** Differential equations can be used to <u>model</u> certain systems where a quantity changes over time. One of the most powerful applications comes when modeling populations

M = CARRYING CAP = 100

$$\left[\frac{dP}{dt} = 2P\left(1 - \frac{P}{100}\right)\right]$$

**Ex 8.** A population is modeled by the differential equation

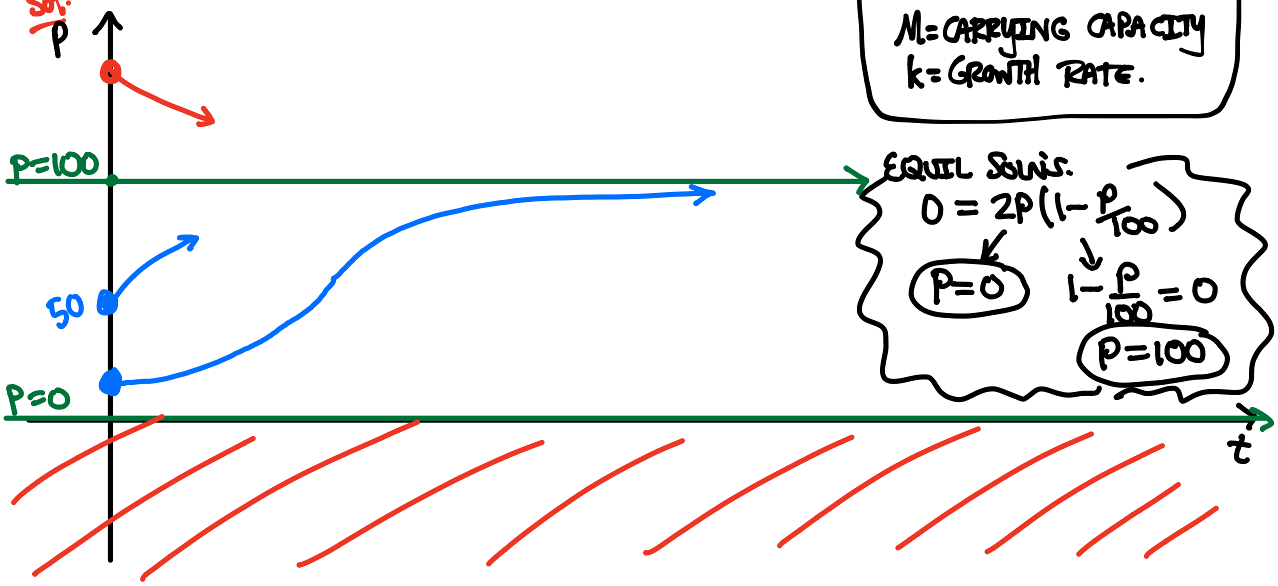
LOGISTIC MODEL:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

M = CARRYING CAPACITY
k = GROWTH RATE.

- $0 < P < 100$ For what values of $P$ is the population INCREASING?
- $P > 100$ For what values of $P$ is the population DECREASING?
- Find all EQUILIBRIUM SOLUTIONS $P = 0$ $P = 100$
- Sketch a plot illustrating this behavior.

Sol:




EQUIL SOLNS.

$$0 = 2P\left(1 - \frac{P}{100}\right)$$

$P = 0$ $\quad 1 - \frac{P}{100} = 0$

$P = 100$