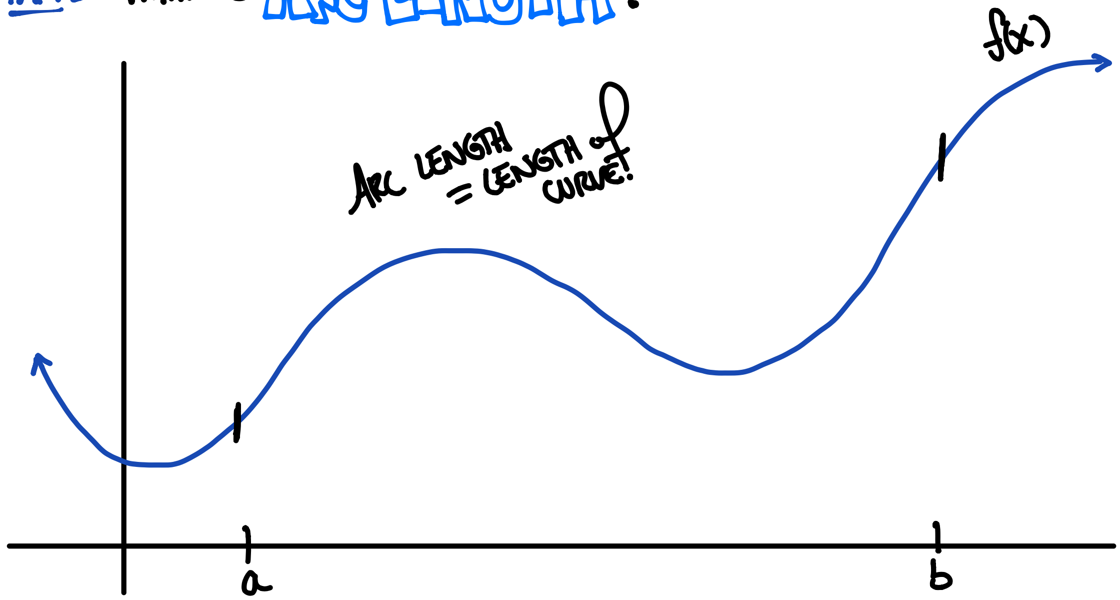
CH 8.1 ARC LENGTH

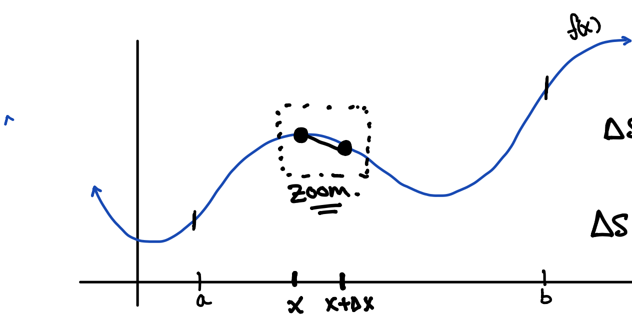
GOAL: Using calculus techniques we will learn how to find the length of certain curves... This is called ARC LENGTH

PART 1: WHAT IS ARC LENGTH?



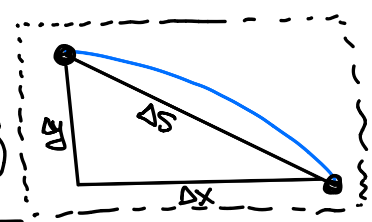
PART 2: HOW CAN WE FIND ARC LENGTH?

* Using calculus techniques, we can derive a formula for ARC LENGTH.



$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\Delta s = \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$



$$\Delta s = \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

slope

Sum all Δs from x=a to x=b & let Δx → 0

$$\boxed{\text{ARC LENGTH } S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

ARC LENGTH FORMULA

If f is continuous on $[a,b]$ then ARC LENGTH of f from $x=a$ to $x=b$ is

① $S = \int_a^b \sqrt{1+(f'(x))^2}\,dx$ IN TERMS of x. $(y=f(x))$.

② $S = \int_c^d \sqrt{1+(f'(y))^2}\,dy$ IN TERMS of y $(x=f(y))$ on $y=c$ to $y=d$

PART 3: SOME EXAMPLES

Ex 1. Find the ARC LENGTH of $y=4\sqrt{x^3}$ on the interval $0\le x\le 1$. Graph the function and indicate the length that was found.

Sol: * IN TERMS of x.

$$S=\int_0^1 \sqrt{1+(f'(x))^2}\,dx$$

$f(x)=4x^{3/2}$

$f'(x)=4\cdot\frac{3}{2}x^{1/2}$

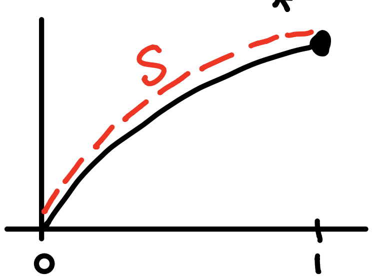
$\boxed{f'(x)=6\sqrt{x}}$

$$S=\int_0^1\sqrt{1+(6\sqrt{x})^2}\,dx$$

$$=\int_0^1\sqrt{1+36x}\,dx$$

$\boxed{w=1+36x \qquad dw/dx=36 \qquad dx=\frac{dw}{36}}$

$$=\int_*^* w^{1/2}\frac{dw}{36}=\frac{1}{36}\frac{w^{3/2}}{3/2}\Big|_*^*=\frac{1}{54}(1+36x)^{3/2}\Big|_0^1$$



$$=\boxed{\frac{1}{54}\left(37^{3/2}-1\right)}$$

Ex 2 Write the ARC LENGTH of each curve (on the specified interval) as a definite integral with respect to the indicated variable.

A $y = \sin(x)$ ON $0 \le x \le \pi$ (w.r.t x) w/ respect to

Sol:

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$$\boxed{S = \int_0^{\pi} \sqrt{1 + (\cos(x))^2}\, dx}$$

B $y = 2x^4$ ON $0 \le x \le 1$ (w.r.t x)

Sol:

$f(x) = 2x^4$

$f'(x) = 8x^3$

$$\boxed{S = \int_0^{1} \sqrt{1 + (8x^3)^2}\, dx}$$

Solve it for x. $\boxed{\frac{y}{2} = x^4 \text{ so } x = \sqrt[4]{\frac{y}{2}}}$

CHANGE TO y. $y = 2x^4$ $x=0 \to y=0$ $x=1 \to y=2$

C $y = 2x^4$ ON $0 \le x \le 1$ (w.r.t y)

Sol: $f(y) = \sqrt[4]{\frac{y}{2}} = \left(\frac{y}{2}\right)^{1/4}$

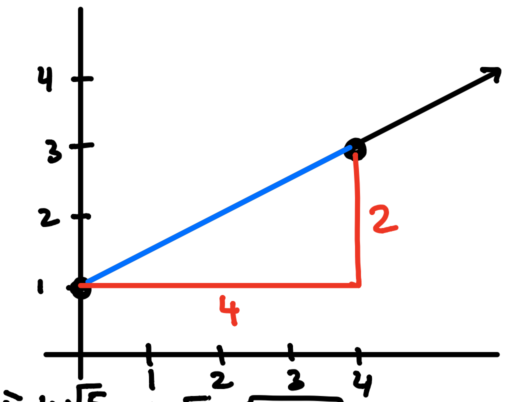
$f'(y) = \frac{1}{4}\left(\frac{y}{2}\right)^{-3/4} \cdot \frac{1}{2} = \frac{1}{8}\left(\frac{y}{2}\right)^{-3/4}$

$$\boxed{S = \int_0^{2} \sqrt{1 + \left(\frac{1}{8}\left(\frac{y}{2}\right)^{-3/4}\right)^2}\, dy}$$

IMPROPER!

Ex 3. Find the ARC LENGTH of the curve $y=\frac{1}{2}x+1$ ON $0 \le x \le 4$.
Verify your answer using geometry!

Sol:



$f(x)=\frac{1}{2}x+1$

$f'(x)=\frac{1}{2}$

$$S=\int_0^4 \sqrt{1+\left(\tfrac{1}{2}\right)^2}\,dx = \int_0^4 \sqrt{\tfrac{5}{4}}\,dx$$

$$= \sqrt{\tfrac{5}{4}}\,x \Big|_0^4 = \boxed{4\sqrt{\tfrac{5}{4}}} = 4\frac{\sqrt{5}}{\sqrt{4}} = \frac{4\sqrt{5}}{2} = \boxed{2\sqrt{5}}$$

*GEOMETRY $\sqrt{4^2+2^2} = \boxed{\sqrt{20}} = \sqrt{4\cdot 5} = \boxed{2\sqrt{5}}$:)

Ex 4. Find the exact length of the curve $y=\frac{x^3}{3}+\frac{1}{4x}$ for $1 \le x \le 3$

Sol: