CH 7.8: IMPROPER INTEGRALS

GOAL: We have learned how to compute the definite integral: $\int_a^b f(x)\,dx$ provided that:

- The limits a and b are finite
- The function f(x) is continuous on [a,b]

If these conditions are not satisfied we have what is called an IMPROPER INTEGRAL and we will learn how to compute them!

REVIEW of TAKING LIMITS

① BASIC FUNCTIONS:

$$\lim_{t\to\infty}\left(\frac{1}{t^{\overset{p>0}{p}}}\right)= \qquad \lim_{t\to\infty}\left(t^{\overset{p>0}{p}}\right)=$$

$$\lim_{t\to\infty}\left(e^{t}\right)= \qquad \lim_{t\to\infty}\left(e^{-t}\right)=$$

$$\lim_{t\to\infty}\left(\ln(t)\right)= \qquad \lim_{t\to\infty}\left(c\right)=$$

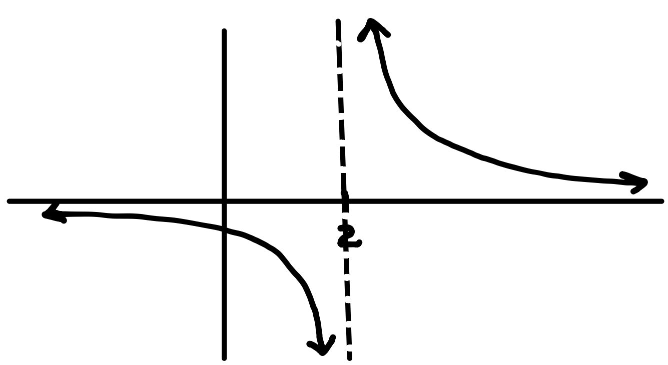
② RATIONAL FUNCTIONS:

- Degree of numerator > degree of denominator $\lim_{t\to\infty}\left(\frac{t^3+2t^2-4}{t^2-8}\right)=$
- Degree of numerator < degree of denominator $\lim_{t\to\infty}\left(\frac{t^2-8}{t^3+2t^2-4}\right)=$
- Degree of numerator = degree of denominator $\lim_{t\to\infty}\left(\frac{t^3+2t^2-4}{4t^3-8}\right)=$

③ ONE-SIDED LIMITS

* A sketch of graph helps!

$$\lim_{t\to 2}\left(\frac{1}{t-2}\right)=$$



④ L'HOPITAL'S RULE (INDETERMINATE FORMS $\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0$, etc)

$$\lim_{t\to\infty}\left(\frac{2t}{e^t}\right) =$$

$$\lim_{t\to\infty}\left(te^{-t}\right) =$$

NOTE: We will treat two types of IMPROPER INTEGRALS $\int_a^b f(x)\,dx$

- TYPE #1: One (or both) of a or b is infinite
- TYPE #2: The function $f(x)$ is discontinuous somewhere on $[a,b]$

PART 1: TYPE #1 IMPROPER INTEGRALS

Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

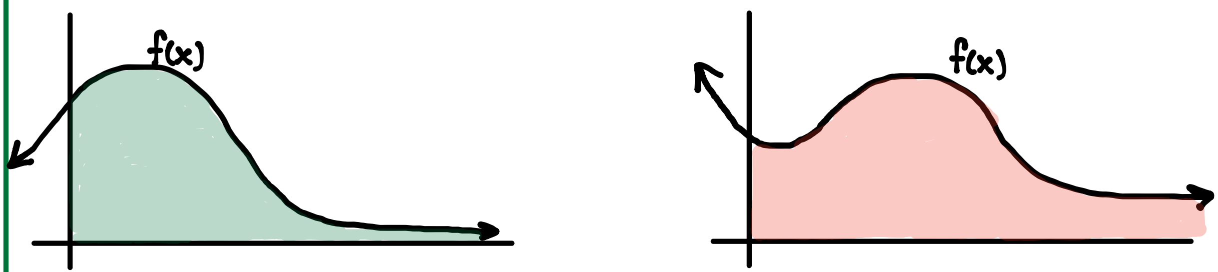
(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used

Interpretation as **AREA:**

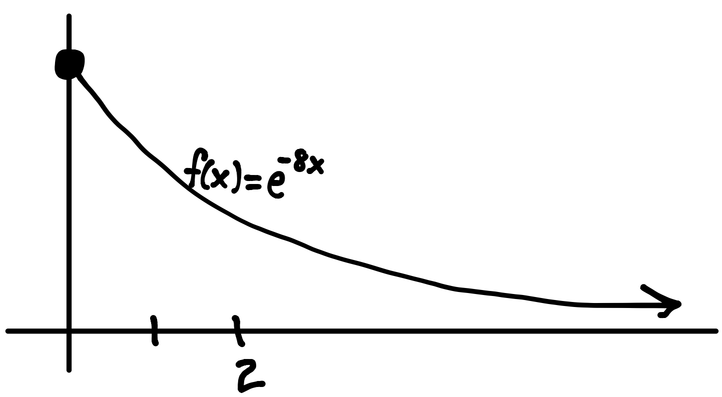
$$\int_0^{\infty} f(x)\,dx$$



Ex 1. Determine if each of the following improper integrals is **CONVERGENT** or **DIVERGENT**. If it is convergent, determine what it converges to.

A $\displaystyle\int_2^{\infty} e^{-8x}\,dx$

sol:

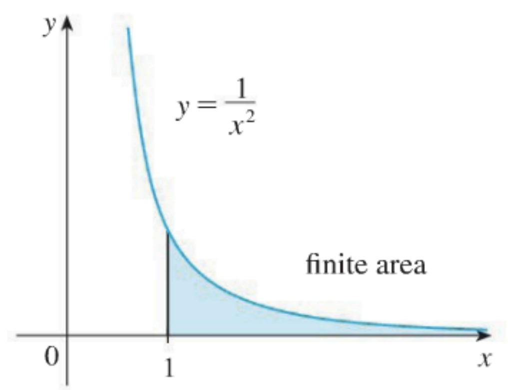


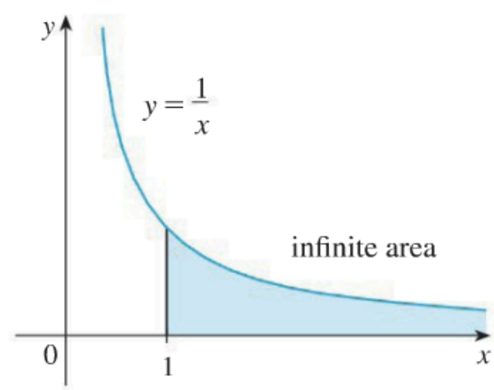
$\boxed{B}$ $\displaystyle\int_a^\infty \frac{1}{x^2}\,dx$ for $a>0$

sol

$\boxed{C}$ $\displaystyle\int_a^\infty \frac{1}{x}\,dx$ for $a>0$

sol:





THM: [P-TEST TYPE 1]

$\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx =$

$\displaystyle\int_1^\infty \frac{4}{\sqrt{x^5}}\,dx$

$\displaystyle\int_1^\infty \frac{1}{x+5}\,dx$

$\boxed{D}$ $\displaystyle\int_2^\infty \frac{dx}{x^2+5x-6}$

sol:

! $\ln(A)-\ln(B)=\ln\left(\frac{A}{B}\right)$
$\ln(A)+\ln(B)=\ln(AB)$

$\boxed{E}$ $\displaystyle\int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx$

sol:

PART 2. TYPE #2 IMPROPER INTEGRALS

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex 2. Determine if each of the following improper integrals is CONVERGENT or DIVERGENT. If it is convergent, determine what it converges to.

$\boxed{A}$ $\displaystyle\int_0^4 \frac{1}{(x-4)^2}\,dx$

Sol:

$\boxed{B}$ $\int_0^1 \frac{1}{x^2}\,dx$

sol:

$\boxed{C}$ $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

sol:

THM: [P-TEST TYPE 2]

$\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx =$

$\int_0^1 \frac{4}{\sqrt{x^5}}\,dx$

$\int_0^1 \frac{1}{x}\,dx$

$\boxed{D}$ $\int_0^2 \ln(x)\,dx$

Let's look at how to treat discontinuities of f(x) that occur inside of the interval [a,b] for $\int_a^b f(x)\,dx$

Ex 3. How would you split up the following improper integral in order to evaluate it?

$$\int_{-2}^{2} \frac{4}{x^2-1}\,dx$$

Sol: