CH 7.8: IMPROPER INTEGRALS

GOAL: We have learned how to compute the definite integral: $\int_a^b f(x)\,dx$ provided that:

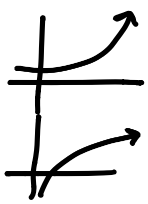
- The limits a and b are finite (1)
- The function f(x) is continuous on [a,b] (2)

If these conditions are not satisfied we have what is called an IMPROPER INTEGRAL and we will learn how to compute them!

REVIEW of TAKING LIMITS

① BASIC FUNCTIONS:

$$\lim_{t\to\infty}\left(\frac{1}{t^p}\right) = 0 \quad p>0 \qquad\qquad \lim_{t\to\infty}\left(t^p\right) = \infty \quad p>0$$



$$\lim_{t\to\infty}\left(e^t\right) = \infty \qquad\qquad \lim_{t\to\infty}\left(e^{-t}\right) = \lim_{t\to\infty}\left(\frac{1}{e^t}\right) = 0$$

$$\lim_{t\to\infty}\left(\ln(t)\right) = \infty \qquad\qquad \lim_{t\to\infty}\left(c\right) = c$$

② RATIONAL FUNCTIONS:

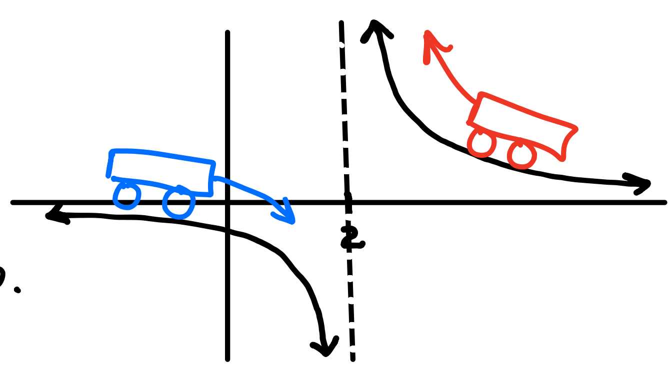
- Degree of numerator > degree of denominator $\quad \lim_{t\to\infty}\left(\dfrac{t^3+2t^2-4}{t^2-8}\right) = \infty$
- Degree of numerator < degree of denominator $\quad \lim_{t\to\infty}\left(\dfrac{t^2-8}{t^3+2t^2-4}\right) = 0$
- Degree of numerator = degree of denominator $\quad \lim_{t\to\infty}\left(\dfrac{t^3+2t^2-4}{4t^3-8}\right) = \dfrac{1}{4}$

③ ONE-SIDED LIMITS

* A sketch of graph helps!

from LEFT $\quad \lim_{t\to 2^-}\left(\dfrac{1}{t-2}\right) = -\infty$

from RIGHT $\quad \lim_{t\to 2^+}\left(\dfrac{1}{t-2}\right) = \infty.$



④ L'HOPITAL'S RULE (INDETERMINATE FORMS $\frac{0}{0}, \frac{\infty}{\infty}, \infty\cdot 0$, etc)

$$\lim_{t\to\infty}\left(\frac{2t}{e^t}\right) \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty}\left(\frac{2}{e^t}\right) = 0$$

LR

$$\lim_{t\to\infty}\left(\underbrace{te^{-t}}_{\infty\cdot 0}\right) = \text{REWrite } \lim_{t\to\infty}\left(\frac{t}{e^t}\right) = \lim_{t\to\infty}\left(\frac{1}{e^t}\right) = 0.$$

LR

NOTE: We will treat two types of IMPROPER INTEGRALS $\int_a^b f(x)\,dx$

- TYPE #1: One (or both) of a or b is infinite
- TYPE #2: The function f(x) is discontinuous somewhere on [a,b]

PART 1: TYPE #1 IMPROPER INTEGRALS

Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

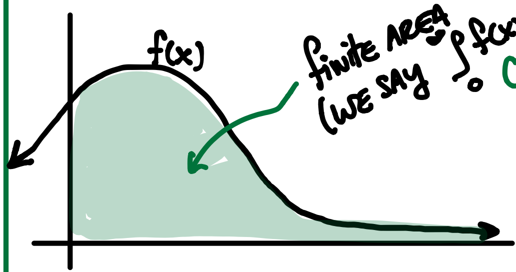
(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

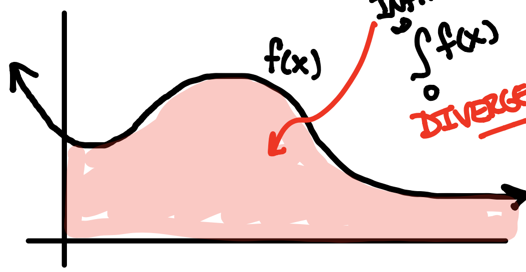
In part (c) any real number a can be used

Interpretation as AREA:

$$\int_0^\infty f(x)\,dx$$



finite AREA (WE SAY $\int_0^\infty f(x)\,dx$ CONVERGES)



Infinite Area $\int_0^\infty f(x)$ DIVERGES

Ex 1. Determine if each of the following improper integrals is CONVERGENT or DIVERGENT. If it is convergent, determine what it converges to.

A $\int_2^\infty e^{-8x}\,dx = \lim_{b\to\infty} \int_2^b e^{-8x}\,dx$

$$\int_2^b e^{-8x}\,dx$$

$$-\frac{1}{8}e^{-8x}\Big|_2^b$$

$$= \left(-\frac{1}{8}e^{-8b} - \left(-\frac{1}{8}e^{-16}\right)\right)$$

$$= \frac{1}{8}\left(e^{-16} - e^{-8b}\right)$$

Sol:

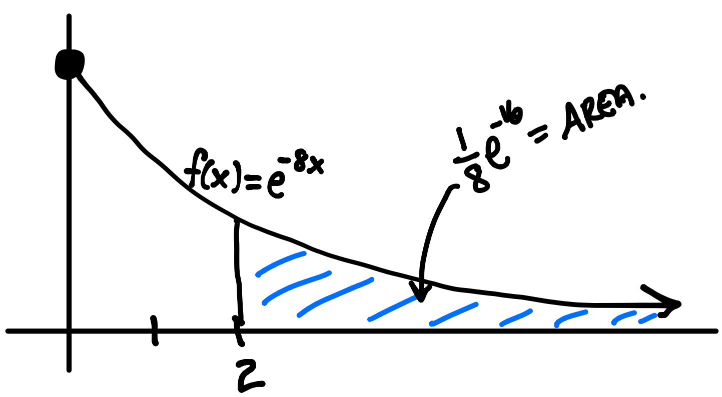
$$= \lim_{b\to\infty}\left[\frac{1}{8}\left(e^{-16} - e^{-8b}\right)\right]$$

$$= \frac{1}{8}\left(e^{-16} - 0\right)$$

$e^{-\infty} = 0$

$e^{\infty} = \infty$

$\int_2^\infty e^{-8x}\,dx$ CONVERGES TO $\rightarrow = \boxed{\frac{1}{8}e^{-16}}$



B $\displaystyle\int_a^\infty \frac{1}{x^2}\,dx$ for $a>0$

Sol $\displaystyle\lim_{b\to\infty}\int_a^b \frac{1}{x^2}\,dx$

$$=\lim_{b\to\infty}\left(\frac{-1}{b}+\frac{1}{a}\right)$$

$$=0+1/a$$

$$\boxed{\begin{array}{l}\displaystyle\int_a^b x^{-2}\,dx\\ \displaystyle -\frac{1}{x}\Big|_a^b\\ \displaystyle\left(\frac{-1}{b}+\frac{1}{a}\right)\end{array}}$$

$\displaystyle\int_a^\infty \frac{1}{x^2}\,dx$ CONVERGES to $1/a$

C $\displaystyle\int_a^\infty \frac{1}{x}\,dx$ for $a>0$

Sol: $\displaystyle\lim_{b\to\infty}\int_a^b \frac{1}{x}\,dx$

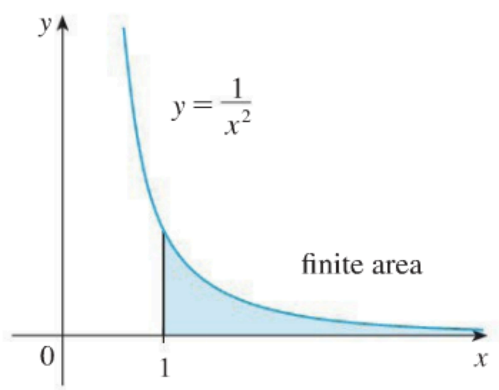
$$=\lim_{b\to\infty}\left(\ln(b)-\ln(a)\right)$$

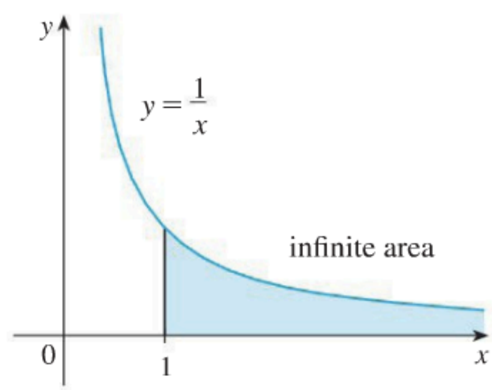
$$=\infty-\ln(a)$$

$$=\infty$$

$$\boxed{\begin{array}{l}\displaystyle\int_a^b \frac{1}{x}\,dx\\ \displaystyle \ln|x|\Big|_a^b\\ \ln(b)-\ln(a)\end{array}}$$

$\displaystyle\int_a^\infty \frac{1}{x}\,dx$ DIVERGES





THM: [P-TEST TYPE 1]

$$\int_a^\infty \frac{1}{x^p}\,dx \quad \begin{cases}\text{CONVERGES if } p>1\\ \text{DIVERGES if } p\le 1\end{cases}$$

for $a>0$

$\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ $p=1/2$ DIVERGES.

$\displaystyle\int_1^\infty \frac{4}{\sqrt{x^5}}\,dx$ $p=5/2$ CONVERGE.

No effect.

$\displaystyle\int_1^\infty \frac{1}{x+5}\,dx=\int \frac{1}{w}\,dw$ $p=1$ DIVERGES.

D $\displaystyle\int_2^{\infty} \frac{dx}{x^2+5x-6} = \lim_{b\to\infty} \underbrace{\int_2^b \frac{dx}{x^2+5x-6}}$

sol:

$\ln(A)-\ln(B)=\ln\left(\frac{A}{B}\right)$
$\ln(A)+\ln(B)=\ln(AB)$

$$\int_2^b \frac{dx}{(x+6)(x-1)} \overset{\text{PARTIAL FRACS}}{=} \int \frac{-1/7}{x+6} + \frac{1/7}{x-1}\,dx$$

$$= -\frac{1}{7}\ln|x+6| + \frac{1}{7}\ln|x-1|$$

$$= \frac{1}{7}\left(\underbrace{\ln|x-1| - \ln|x+6|}_{\text{LOG RULE.}}\right)$$

$$= \frac{1}{7}\left(\ln\left|\frac{x-1}{x+6}\right|\right)\Bigg|_2^b$$

$$= \frac{1}{7}\left[\ln\left(\frac{b-1}{b+6}\right) - \ln(1/8)\right]$$

$$\frac{1}{7}\lim_{b\to\infty}\left(\left[\ln\left(\frac{b-1}{b+6}\right) - \ln(1/8)\right]\right)$$

$$= \frac{1}{7}\lim_{b\to\infty}\left(\ln\left(\frac{b-1}{b+6}\right)\right) - \frac{1}{7}\lim\left(\underline{\ln\left(\frac{1}{8}\right)}\right)$$

$$= \frac{1}{7}\ln\left(\underbrace{\lim_{b\to\infty}\left(\frac{b-1}{b+6}\right)}\right) - \frac{1}{7}\ln\left(\frac{1}{8}\right)$$

$$= \frac{1}{7}\cancel{\ln(1)}^{0} - \frac{1}{7}\ln\left(\frac{1}{8}\right) = \boxed{-\frac{1}{7}\ln\left(\frac{1}{8}\right)}$$

CONVERGES

! If f(x) is CONTINUOUS
$\lim[f(g(x))] = f(\lim(g(x)))$

E $\displaystyle\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,dx = \lim_{t\to-\infty} \underbrace{\int_t^0 \frac{e^x}{1+e^x}\,dx}$

sol:

$$\int_t^0 \frac{e^x}{1+e^x}\,dx \qquad w = 1+e^x \qquad dw/dx = e^x \qquad dx = \frac{dw}{e^x}$$

$$= \int_*^* \frac{\cancel{e^x}}{w}\frac{dw}{\cancel{e^x}}$$

$$= \ln|w|$$

$$= \ln|1+e^x|\Big|_t^0$$

$$= \ln(2) - \ln(1+e^t)$$

$$= \boxed{\ln(2) - \ln(1+e^t)}$$

$$= \lim_{t\to-\infty}\left(\underline{\ln(2)} - \underline{\ln(1+e^t)}\right)$$

$$\ln(2) - \ln\left(\lim_{t\to-\infty}(1+\underbrace{e^t}_{0})\right)$$

$$= \ln(2) - \cancel{\ln(1)}^{0}$$

$$= \boxed{\ln(2)} \quad \text{CONVERGES.}$$

! If BOTH BOUNDS ARE INFINITE.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

* if 1 DIVERGES THEN THE INTEGRAL DIVERGES.

PART 2. TYPE #2 IMPROPER INTEGRALS

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

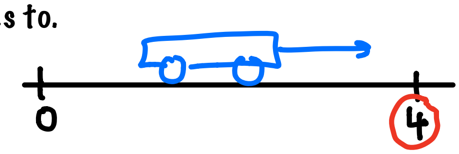
The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex 2. Determine if each of the following improper integrals is CONVERGENT or DIVERGENT. If it is convergent, determine what it converges to.

A $\displaystyle\int_0^4 \frac{1}{(x-4)^2}\,dx$ $x \neq 4$



Sol: $\displaystyle = \lim_{b \to 4^-} \int_0^b \frac{1}{(x-4)^2}\,dx$

$$\int_0^b \frac{1}{(x-4)^2} \qquad w = x-4, \quad dw = dx$$

$$\int_*^* w^{-2}\,dw = \frac{-1}{w}\Big|_*^*$$

$$\frac{-1}{x-4}\Big|_0^b = \left(\frac{-1}{b-4} - \frac{-1}{-4}\right)$$

$$= \lim_{b \to 4^-} \left[-\left(\frac{1}{b-4} + \frac{1}{4} \right) \right]$$

$$= -\lim_{b \to 4^-} \left(\frac{1}{b-4} + \frac{1}{4} \right)$$

$$= -\left(-\infty + \frac{1}{4} \right)$$

$$= \boxed{\infty}$$

DIVERGES.

B $\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \underbrace{\int_t^1 x^{-2}\,dx}_{\text{integrate}}$

Sol:

$$\lim_{t\to 0^+}\left(-1 + \frac{1}{t}\right)$$

$$-1 + \infty$$

$$= \boxed{\infty} \quad \text{DIVERGES.}$$

C $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx$

Sol:

$$\lim_{t\to 0^+}\left(2 - 2\sqrt{t}\right)$$

$$= 2 + 0.$$

$$= \boxed{2} \quad \text{CONVERGES.}$$

THM: [P-TEST TYPE 2]

$$\int_0^1 \frac{1}{x^p}\,dx \quad \begin{cases} \text{CONVERGES} & \text{if } p < 1 \\ \text{DIVERGES} & \text{if } p \geq 1 \end{cases}$$

$\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$ $p=1/3$ CONVERGE

$\int_0^1 \frac{4}{\sqrt{x^5}}\,dx$ $p=5/2$ DIVERGE

$\int_0^1 \frac{1}{x}\,dx$ $p=1$ DIVERGE.

D $\int_0^2 \ln(x)\,dx = \lim_{t\to 0^+} \int_t^2 \ln(x)\,dx.$

$$= \lim_{t\to 0^+}\left(t - t\ln(t) + 2\ln(2) - 2\right)$$

$$= 0 - \lim_{t\to 0^+}\left(t\ln(t)\right) + 2\ln(2) - 2$$

$0\cdot\infty$

Rewrite $= 0 - \lim_{t\to 0^+}\left(\frac{\ln(t)}{1/t}\right) + 2\ln(2) - 2$

$\int_t^2 \ln(x)\,dx$

$u = \ln(x)$
$du = \frac{dx}{x}$
$dv = dx$
$v = x$

$x\ln(x) - \int x \frac{dx}{x}$

$= x\ln(x) - \int 1\,dx$

$= x\ln(x) - x \Big|_t^2$

$= (2\ln(2) - 2) - (t\ln(t) - t)$

$= t - \ln(t) + 2\ln(2) - 2$

$\downarrow$ LR.

$$= -\lim_{t\to 0^+}\left(\frac{1/t}{-1/t^2}\right) + 2\ln(2) - 2$$

$$= -\lim_{t\to 0^+}(-t) + 2\ln(2) - 2$$

$$= 0 + 2\ln(2) - 2$$

CONVERGES.

Let's look at how to treat discontinuities of f(x) that occur <u>inside</u> of the interval [a,b] for

$$\int_a^b f(x)\,dx$$

Ex 3. How would you split up the following improper integral in order to evaluate it?

$$\int_{-2}^{2} \overbrace{\frac{4}{x^2-1}}^{f(x)}\,dx$$

$x \neq 1$ & $x \neq -1$

Sol:

$$= \int_{-2}^{-1} f(x)\,dx + \underbrace{\int_{-1}^{1} f(x)\,dx}_{\text{split again.}} + \int_{1}^{2} f(x)\,dx$$

$$\boxed{\int_{-2}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx.}$$