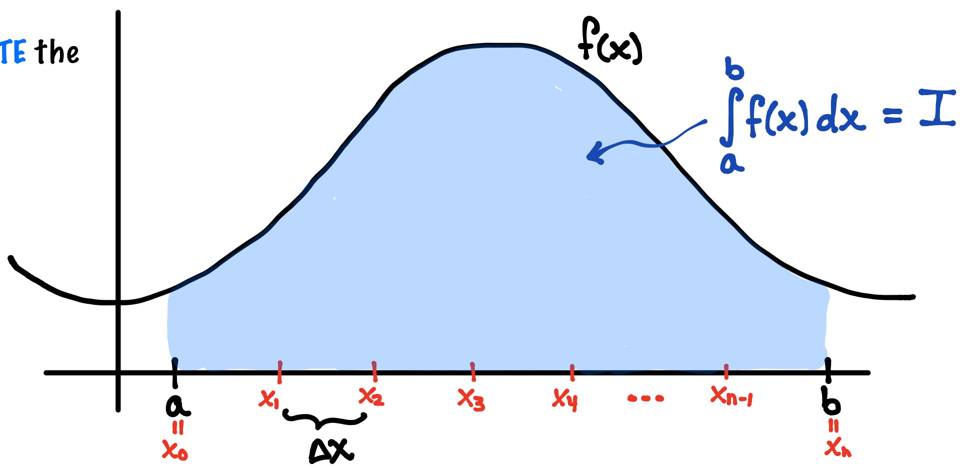
CH 7.7: APPROXIMATE INTEGRATION

MOTIVATION: The DEFINITE INTEGRAL $\int_a^b f(x)\,dx$ represents the signed area between the graph of the function f(x) and the x-axis (positive area above x-axis and negative below). Oftentimes, our integration methods DO NOT WORK, so we need to resort to APPROXIMATION TECHNIQUES.

PART 1: THE 5 METHODS:

* We want to APPROXIMATE the blue area!

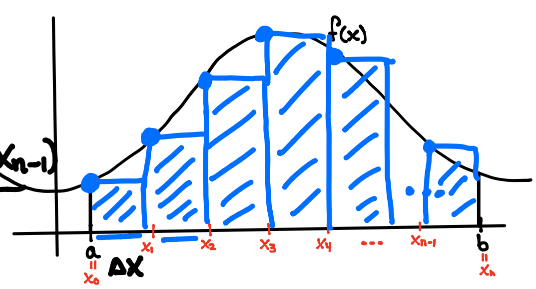


Number of intervals: n Width of each interval: $\Delta x = \frac{b-a}{n}$

METHOD #1: [LEFT-HAND SUM] "L_n"

n ← # of RECTANGLES.

$$L_n = \underbrace{\Delta x \cdot f(x_0)}_{\text{REC \#1}} + \underbrace{\Delta x \cdot f(x_1)}_{\text{REC \#2}} + \dots + \underbrace{\Delta x \cdot f(x_{n-1})}_{\text{REC \#n}}$$

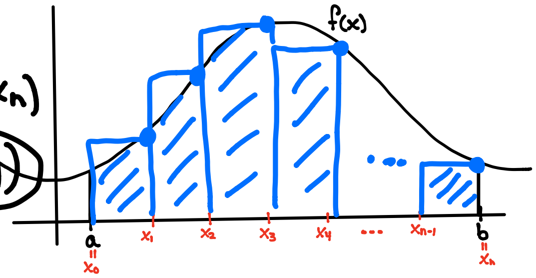


$$L_n = \Delta x\,\underbrace{\left(f(x_0) + f(x_1) + \dots + f(x_{n-1})\right)}_{n}$$

METHOD #2: [RIGHT-HAND SUM] "R_n"

$$R_n = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \dots + \Delta x \cdot f(x_n)$$

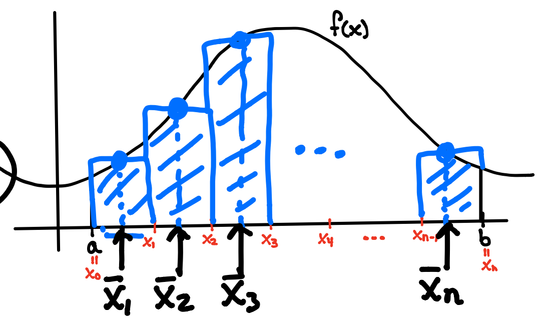
$$R_n = \Delta x\,\underbrace{\left(f(x_1) + f(x_2) + \dots + f(x_n)\right)}_{n}$$



METHOD #3: [MIDPOINT RULE] "M_n"

$$\bar{x}_j = \frac{x_j + x_{j-1}}{2}$$

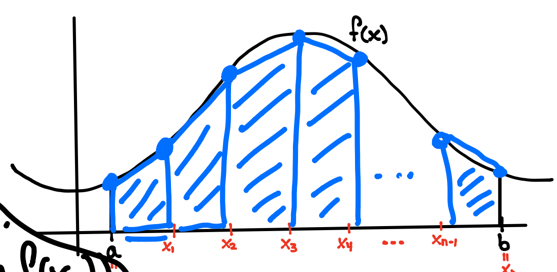
$$M_n = \Delta x \cdot \left(f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\right)$$



METHOD #4: [TRAPEZOIDAL RULE] "T_n"

$$T_n = \frac{L_n + R_n}{2}$$

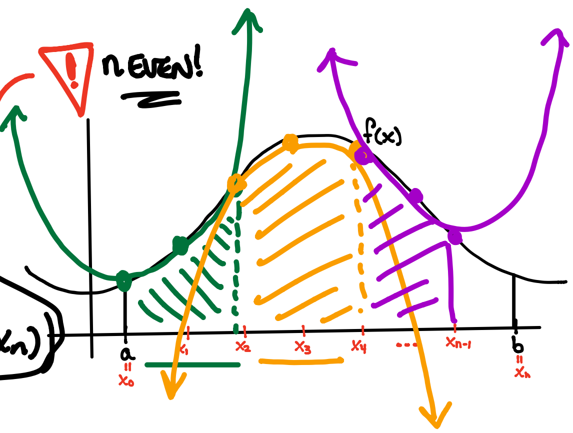
$$T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right)$$



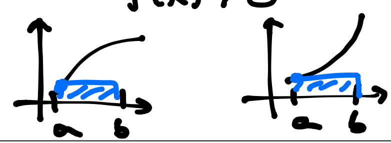
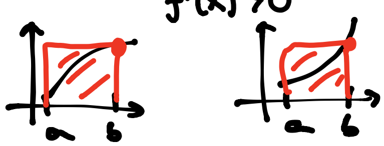
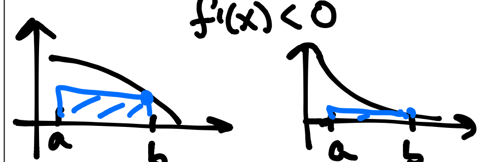
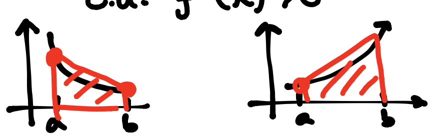
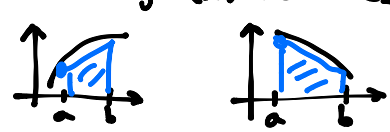
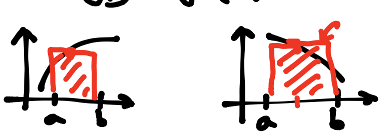
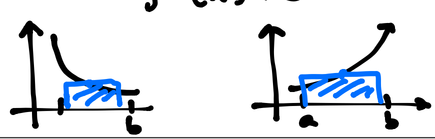
METHOD # 5: [SIMPSON'S RULE] "S_n"

⚠ n EVEN!

$$S_n = \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots 4f(x_{n-1}) + f(x_n)\Big)$$



PART 2: OVER or UNDER?

Rule	Overestimate of $\int_a^b f(x)dx$ when...	Underestimate of $\int_a^b f(x)dx$ when...
LEFT L_n n=1	$f'(x) < 0$ 	$f'(x) > 0$ 
RIGHT R_n	$f'(x) > 0$ 	$f'(x) < 0$ 
TRAP T_n	C.U. $f''(x) > 0$ 	$f''(x) < 0$ CD 
MID M_n	CD $f''(x) < 0$ 	CU $f''(x) > 0$ 

PART 3: ERROR IN APPROXIMATIONS of $\int_a^b f(x)\,dx$

ERROR IN TRAP.

$$E_T = \int_a^b f(x)\,dx - T_n$$

EXACT — APPROX.

ERROR IN MIDPT

$$E_M = \int_a^b f(x)\,dx - M_n$$

ERROR w/ SIMP.

$$E_S = \int_a^b f(x)\,dx - S_n$$

THM: [ERROR BOUND for TRAP and MID]

Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the TRAPEZOIDAL RULE and MIDPOINT RULE, then:

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

BIGGEST ERROR COULD BE. "ERROR BOUND"

BOUND ERROR

THM: [ERROR BOUND for SIMPSON'S RULE]

Suppose $|f^{(4)}(x)| \le K$ for $a \le x \le b$. If E_S is the error involved in using SIMPSON'S RULE, then:

4th DER.

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

NOTE:

- n INCREASING LOWERS BOUND.
- K IS CALLED UPPER BOUND ON $|f''(x)|$ or $|f^{(4)}(x)|$.
- MAX MIDPT ERROR IS ½ THAT of TRAP.

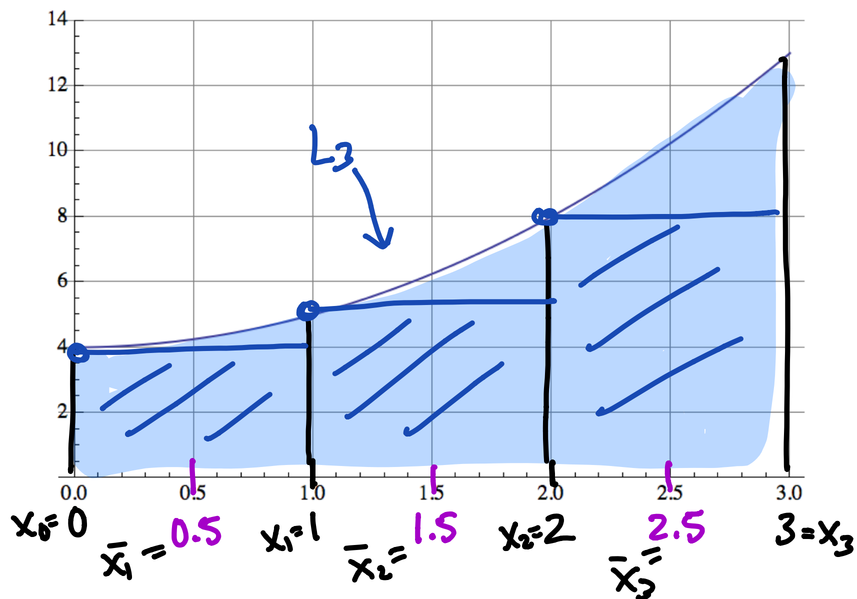
PART 4: EXAMPLES

Ex 1. Given the graph of f(x), let $I=\int_0^3 f(x)\,dx$ and find the following approximations of I. Also label each as an over or under estimate:

$n=3$

$\Delta x=\frac{b-a}{n}=\frac{3-0}{3}$

$\Delta x=1$



x	f(x)
0	4
0.5	4.25
1	5
1.5	6.25
2	8
2.5	10.25
3	13

A L_3

$=\Delta x\left(f(x_0)+f(x_1)+f(x_2)\right)=1(4+5+8)$
$=\boxed{17}$

B $R_3=\Delta x\left(f(x_1)+f(x_2)+f(x_3)\right)$
$=1(5+8+13)=\boxed{26}$

C $M_3=\Delta x\left(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)\right)=1(4.25+6.25+10.25)$
$=\boxed{20.75}$

D $T_3=\frac{L_3+R_3}{2}=\frac{17+26}{2}=\boxed{21.5}$

!

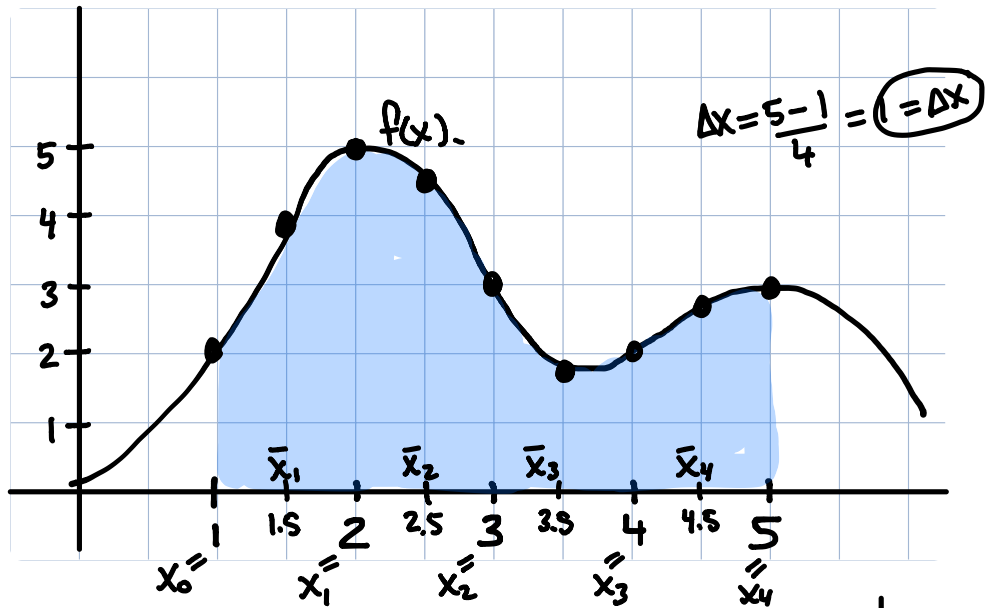
$f(x)=x^2+4$

$\int_0^3 x^2+4\,dx$
$=\frac{x^3}{3}+4x\Big|_0^3$
$=\boxed{21}$

List I, R_3, L_3, M_3, T_3 in order from least to greatest:

$L_3<M_3<I<T_3<R_3$

Ex 2: Use the TRAPEZOIDAL RULE, SIMPSON'S RULE, and MIDPOINT RULE with n=4 to approximate the area under the graph shown below:



sol:

$$T_4 = \frac{L_4+R_4}{2} = \frac{\Delta x}{2}\big(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\big)$$

$$= \frac{1}{2}(2+10+6+4+3) = \boxed{12.5}$$

$$M_4 = \Delta x\big(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+f(\bar{x}_4)\big)$$

$$= 1\cdot(4+4.5+1.75+2.75) = \boxed{13}$$

$$S_4 = \frac{\Delta x}{3}\big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\big)$$

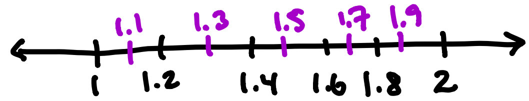
$$= \frac{1}{3}(2+4(5)+2(3)+4(2)+3) = \frac{1}{3}(2+20+6+8+3)$$

$$= \boxed{13}$$

x	f(x)
1	2
1.5	4
2	5
2.5	4.5
3	3
3.5	1.75
4	2
4.5	2.75
5	3

Ex 3. Estimate the value of the definite integral $\int_1^2 (1/x)\,dx$ using n=5 and the following approximation methods:

$f(x)$



A L_5

B R_5

C M_5

D T_5

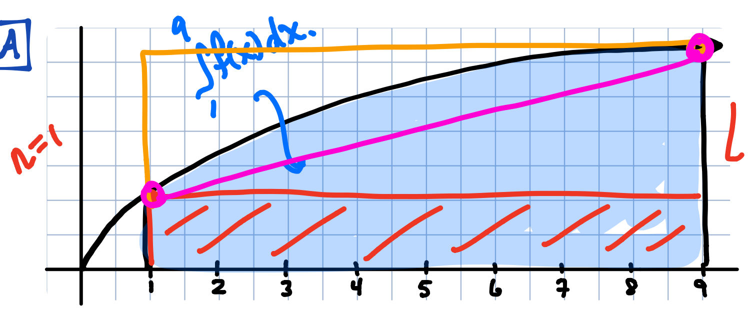
E S_5

x	f(x)
1	.
1.1	
1.2	
1.3	
1.4	
1.5	
1.6	
1.7	
1.8	
1.9	
2	

Ex 4. For each f(x) drawn below, list $\int_1^9 f(x)\,dx, T_8, M_8, R_8, L_8$ in order from smallest to largest.

CONCAVE DWN , ↑

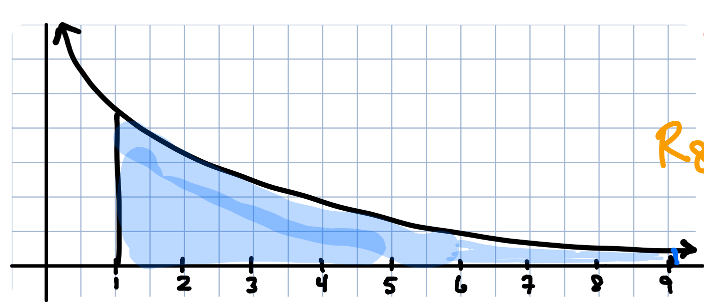
A



sol:

$$L_8 \le T_8 \le \int_1^9 f(x)\,dx \le M_8 \le R_8$$

B



sol:

$$R_8 \le M_8 \le \int_1^9 f(x)\,dx \le T_8 \le L_8$$

Ex 5. How large should we take "n" in order to guarantee that the TRAPEZOIDAL RULE and MIDPOINT RULE approximations for $\int_1^2 1/x\,dx$ are accurate to within 0.0001?

Sol: $|E_T| \le \dfrac{k(b-a)^3}{12n^2}$ w/ $|f''(x)| \le k$ on $[a,b]$

$a=1$ $b=2$

$f(x) = 1/x$.

STEP 1: SET ACCURACY:

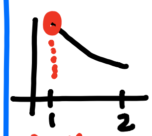
$$|E_T| \le \frac{k(b-a)^3}{12n^2} < 0.0001$$

STEP 2: FIND "k". $f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$ $|f''(x)| = \dfrac{2}{x^3}$ ~DECREASING.

* k IS MAX of $|f''(x)|$ ON $[1,2]$



$k = \dfrac{2}{1^3} = 2 = k$

STEP 3: INPUT k, a, b TO STEP 1.

$$\frac{k(b-a)^3}{12n^2} < 0.0001$$

$$\frac{2(2-1)^3}{12n^2} < 0.0001 \Rightarrow \frac{1}{6n^2} < 0.0001$$

STEP 4: SOLVE for "n" * TREAT < LIKE =

$$\frac{1}{6n^2} < 0.0001$$

$$\Rightarrow \frac{1}{0.0001} < 6n^2$$

$$n^2 > \frac{1}{0.0006} \Rightarrow n > \sqrt{\frac{1}{0.0006}} = 40.8$$

$n = 41$

Ex 6. How large should "n" be to guarantee that the SIMPSON'S RULE approximation of $\int_0^1 4e^{x^2}\,dx$ is accurate to within 0.0001?

Sol: $|E_S| \le \dfrac{k(b-a)^5}{180n^4}$ $|f^{(4)}(x)| \le k$ on $[0,1]$

STEP 1: SET ACCURACY:

$$\frac{k(b-a)^5}{180n^4} < 0.0001$$

STEP 2: FIND "k". $f(x) = 4e^{x^2}$

f', f'', f'''

$f^{(4)}(x) = 16e^{x^2}(4x^4 + 2x^2 + 3)$

$|f^{(4)}(x)| = 16e^{x^2}(4x^4 + 12x^2 + 3)$

* k IS MAX of ↗ ON $[0,1]$

$k = 16e^1(4\cdot1 + 2\cdot1 + 3) = 304e$

STEP 3: INPUT k, a, b TO STEP 1.

$$\frac{k(b-a)^5}{180n^4} < 0.0001$$

$$\frac{304e\cdot1}{180n^4} < 0.0001$$

STEP 4: SOLVE for "n"

$$304e < 0.0001\cdot180n^4$$

$$\sqrt[4]{\frac{304e}{0.0001\cdot180}} < n$$

$\Rightarrow n > 14.63$ SO WE NEED $n = 16$

! n must BE EVEN