CH 7.4 PARTIAL FRACTIONS

METHOD of INTEGRATION.

MOTIVATION: Compute the following indefinite integrals:

A $\int \frac{4}{x} + \frac{2}{x+1}\,dx$

sol: $= 4\int \frac{1}{x}\,dx + 2\int \frac{1}{x+1}\,dx$

$\boxed{= 4\ln|x| + 2\ln|x+1| + C}$

B $\int \frac{6x+4}{x^2+x}\,dx$

sol: EQUAL

$\int \left(\frac{4}{x} + \frac{2}{x+1}\right) dx = \boxed{4\ln|x| + 2\ln|x+1| + C}$

!

$$\frac{4}{x} + \frac{2}{x+1} = \frac{4(x+1) + 2x}{x(x+1)} = \frac{6x+4}{x^2+x}.$$

ALGEBRA.

PARTIAL FRAC DECOMP.

GOAL: We want to be able to take an expression $\frac{6x+4}{x^2+x}$

and rewrite it as $\frac{4}{x} + \frac{2}{x+1}$

in order to more easily integrate! This is called the METHOD of PARTIAL FRACTION DECOMPOSITION

REVIEW: Some things you ought to master before we begin:

① LOG INTEGRALS: $\int \frac{1}{x}\,dx = \ln|x| + C$

Ex. A $\int \frac{1}{x+4}\,dx$

$w = x+4$
$dw/dx = 1$
$dw = dx$

$\int \frac{1}{w}\,dw = \ln|w| + C$

$= \boxed{\ln|x+4| + C}$

B $\int \frac{1}{4-x}\,dx = \boxed{-\ln|4-x| + C}$

C $\int \frac{1}{2x+1}\,dx$

$w = 2x+1$
$\frac{dw}{dx} = 2$
$dx = \frac{dw}{2}$

$= \int \frac{1}{w}\,\frac{dw}{2}$

$= \boxed{\frac{1}{2}\ln|2x+1| + C}$

D $\int \frac{1}{5-3x}\,dx = \boxed{-\frac{1}{3}\ln|5-3x| + C}$

② FACTORING: ALWAYS check your factoring (by expanding it back out) before you proceed with the integration!

③ ARCTAN INTEGRALS: $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$ for $a \neq 0$

Ex. A $\int \frac{1}{x^2+25}\,dx$ $a=5$

$= \boxed{\frac{1}{5}\arctan\left(\frac{x}{5}\right)+C}$

B $\int \frac{6}{4x^2+9}\,dx = 6\int \frac{1}{(2x)^2+9}\,dx$

$\boxed{w=2x \quad \frac{dw}{dx}=2 \quad dx=\frac{dw}{2}}$

$= 6\int \frac{1}{w^2+9}\frac{dw}{2} = 3\int \frac{1}{w^2+9}\,dw$

$= 3\cdot\frac{1}{3}\arctan\left(\frac{w}{3}\right) = \boxed{\arctan\left(\frac{2x}{3}\right)+C}$

PART 1: THE PROCEDURE

* We will consider integrals involving RATIONAL EXPRESSIONS of the form P(x)/Q(x), where P(x) and Q(x) are Polynomials (with the degree of P(x) less than the degree of Q(x))

STEP 1: FACTOR the denominator fully

STEP 2: Determine the FORM of the PARTIAL FRACTION decomposition depending on the types of factors in denominator. We will deal with three cases:

Parameters

- Distinct LINEAR Factors:
  DENOM $= (x+a)(x+b)$ Form $\frac{A}{x+a}+\frac{B}{x+b}$

  *CAN BE MORE TERMS. $\frac{C}{x+c}+\frac{D}{x+d}\cdots$

- Distinct, irreducible QUADRATIC Factors:
  Cannot factor.
  DENOM $= ax^2+bx+c$ Form: $\frac{Ax+B}{ax^2+bx+c}$

- Repeated LINEAR factors:
  DENOM $= (x+a)^k$ Form: $\frac{A}{x+a}+\frac{B}{(x+a)^2}+\cdots\frac{Z}{(x+a)^k}$

STEP 3: Solve for the UNKNOWN PARAMETERS. Multiply both sides of equation by the denominator, then make judicious choices for x (see example!)

*Fancy ALGEBRA.

STEP 4: With the new simplified form, you can proceed to INTEGRATE. If all went accordingly you should be able to apply substitution or use arctan rule.

PART 2: SOME EXAMPLES

Ex 1. Compute the following integrals using PARTIAL FRACTION DECOMPOSITION.

A $\int \frac{6x+4}{x^2+x}\,dx$

ALGEBRA

Sol:

$$\frac{6x+4}{x^2+x} = \frac{6x+4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

linear   linear

$$\cancel{x(x+1)}\frac{6x+4}{\cancel{x(x+1)}} = x(x+1)\left(\frac{A}{x} + \frac{B}{x+1}\right)$$

$$6x+4 = A(x+1) + Bx$$

$\boxed{x=0}$ $\quad 4 = A + 0 \quad A=4$

$\boxed{x=-1}$ $\quad -6+4 = 0 + B(-1) \quad B=2$

MANY METHODS from THIS pt.

METHOD 1: CHOOSE SPECIAL x VALUES.

$$\int \frac{6x+4}{x^2+x}\,dx = \int \frac{4}{x} + \frac{2}{x+1}\,dx$$

$$= 4\int \frac{1}{x}\,dx + 2\int \frac{1}{x+1}\,dx$$

$$= \boxed{4\ln|x| + 2\ln|x+1| + C}$$

CALCULUS

B $\int \frac{11x-2}{x^2-x-12}\,dx$

Sol:

$$\frac{11x-2}{x^2-x-12} = \frac{11x-2}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$$

$$\cancel{(x-4)(x+3)}\frac{11x-2}{\cancel{(x-4)(x+3)}} = (x-4)(x+3)\left(\frac{A}{x-4} + \frac{B}{x+3}\right)$$

$$11x-2 = A(x+3) + B(x-4)$$

$\boxed{x=-3}$ $\quad -35 = 0 + B(-7) \quad B=5$

$\boxed{x=4}$ $\quad 42 = 7A + 0 \quad A=6$

$$\int\left(\frac{6}{x-4} + \frac{5}{x+3}\right)dx$$

$$6\int \frac{1}{x-4}\,dx + 5\int \frac{1}{x+3}\,dx$$

$$\boxed{6\ln|x-4| + 5\ln|x+3| + C}$$

$\boxed{C}$ $\displaystyle\int \frac{2x^2-3x+8}{x^3+4x}\,dx \longrightarrow \int \left(\frac{2}{x} - \frac{3}{x^2+4}\right)dx$

Sol:

$$\frac{2x^2-3x+8}{x^3+4x} = \frac{2x^2-3x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

linear   irr. quad

$$2x^2-3x+8 = A(x^2+4)+(Bx+C)\cdot x$$

$$2x^2-3x+8 = Ax^2+4A+Bx^2+Cx$$

$$2\int \frac{1}{x}\,dx - 3\underbrace{\int \frac{1}{x^2+4}\,dx}_{\text{arctan.}}$$

$$2\ln|x| - 3\cdot\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

! METHOD 2: "EQUATING COEFFICIENTS"

$$\boxed{2}\cancel{x^2}\boxed{-3}\cancel{x}+\boxed{8} = \boxed{(A+B)}\cancel{x^2} + \boxed{C}\cancel{x} + \boxed{4A}$$

$A+B=2$ $B=0$

$C=-3$ $C=-3$

$8=4A$ $A=2$

$\boxed{D}$ $\int \frac{2x^2+17x-12}{x^3+5x^2-6x}\,dx \longrightarrow \int \frac{2}{x} + \frac{-1}{x+6} + \frac{1}{x-1}\,dx$

$= \boxed{2\ln|x| - \ln|x+6| + \ln|x-1| + C}$

sol:

$$\frac{2x^2+17x-12}{x(x+6)(x-1)} = \frac{A}{x} + \frac{B}{x+6} + \frac{C}{x-1}$$

$$2x^2+17x-12 = A(x+6)(x-1) + Bx(x-1) + Cx(x+6)$$

$\boxed{x=1}$ $\quad 7 = 7C \quad C=1$

$\boxed{x=0}$ $\quad -12 = -6A \quad A=2$

$\boxed{x=-6}$ $\quad -42 = 42B \quad B=-1$

$\boxed{E}$ $\int \frac{2}{x^3-9x}\,dx$

sol:

$$\frac{2}{x^3-9x} = \frac{2}{x(x^2-9)} = \frac{2}{x(x-3)(x+3)}$$

F $\displaystyle\int \frac{x+3}{x^2+2x+1}\,dx \longrightarrow \int \frac{1}{x+1}\,dx + \int \frac{2}{(x+1)^2}\,dx$

Sol:

$$\frac{x+3}{x^2+2x+1} = \frac{x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Repeated Linear.

$$x+3 = A(x+1) + B$$

$$\boxed{1}\cancel{x}+\boxed{3} = \boxed{A}\cancel{x} + \boxed{A+B}$$

$A=1$ $A=1$

$A+B=3$ $B=2$

$w = x+1$
$dw = dx$

$$= \ln|x+1| + \int \frac{2}{w^2}\,dw$$

$$= \ln|x+1| + 2\int w^{-2}\,dw$$

$$= \ln|x+1| + \frac{2w^{-1}}{-1} + C$$

$$= \boxed{\ln|x+1| - \frac{2}{x+1} + C}$$

NOTE: We can treat cases where the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$ by using polynomial long division. This is beyond the scope of the course.

PART 3: SMARTER NOT HARDER

Ex 2. Compute the following integrals:

A $\displaystyle\int \frac{dx}{x^2+4x+7}$

Sol:

!

$$\cancel{\frac{1}{x^2+4x+7} = \frac{Ax+B}{x^2+4x+7} = \frac{0x+1}{x^2+4x+7}}$$

$$\cancel{Ax+B=1 \quad B=1 \quad A=0}$$

COMPLETE THE SQUARE: $\left(\frac{4}{2}\right)^2$

$$\underbrace{x^2+4x+4}_{(x+2)^2} - 4 + 7 = (x+2)^2+3$$

$$\int \frac{dx}{x^2+4x+7} = \int \frac{dx}{(x+2)^2+3}$$

$w = x+2$
$dw = dx$

$$= \int \frac{dw}{w^2+3}$$

ARCTAN w/ $a=\sqrt{3}$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{w}{\sqrt{3}}\right) + C = \boxed{\frac{1}{\sqrt{3}}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C}$$

$\boxed{B}$ $\displaystyle\int \frac{2x+2}{x^2+2x+5}\,dx$

$$\boxed{\begin{aligned} w &= x^2+2x+5 \\ \frac{dw}{dx} &= 2x+2 \\ dx &= \frac{dw}{2x+2} \end{aligned}}$$

sol:

$$\int \frac{\cancel{2x+2}}{w}\,\frac{dw}{\cancel{2x+2}}$$

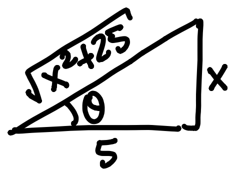
$$= \ln|w| + c$$

$$= \boxed{\ln|x^2+2x+5| + C}$$

EXTRA:

$$\int \frac{dx}{x^2+25}$$

* USE TRIG SUBST.



$\theta = \arctan(x/5)$

$\tan\theta = \frac{x}{5}$ $\quad x = 5\tan\theta$

$dx = 5\sec^2\theta\, d\theta$

$\sec\theta = \frac{\sqrt{x^2+25}}{5}$

$\sqrt{x^2+25} = 5\sec\theta.$

$$= \int \frac{dx}{(\sqrt{x^2+25})^2}$$

$$\int \frac{5\sec^2\theta\, d\theta}{(5\sec\theta)^2} = \int \frac{5\cancel{\sec^2\theta}\, d\theta}{25\cancel{\sec^2\theta}}$$

$$= \frac{1}{5}\int d\theta = \frac{1}{5}\theta + C$$

$$= \boxed{\frac{1}{5}\arctan\left(\frac{x}{5}\right) + c}$$