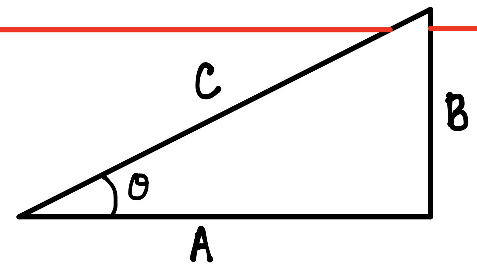
CH 7.3 TRIG SUBSTITUTION

We will now learn a method to treat more complicated integrals for which normal substitution will not work immediately (always check!). This method involves using TRIGONOMETRY to make a creative substitution within an integral.

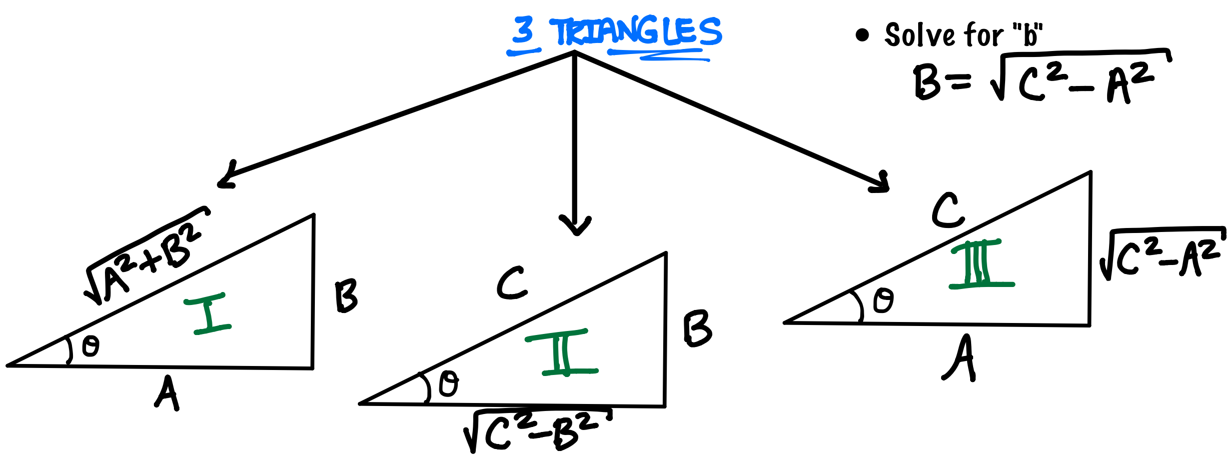
REVIEW:



PYTHAGOREAN THEOREM

$$A^2 + B^2 = C^2$$

- Solve for "c"
  $C = \sqrt{A^2+B^2}$
- Solve for "a"
  $A = \sqrt{C^2 - B^2}$
- Solve for "b"
  $B = \sqrt{C^2 - A^2}$



NOTE: Cosecant, secant, and cotan are reciprocals!

** SOH.CAH.TOA

$$\boxed{\sin(\theta) = \frac{OPP}{HYP}, \quad \cos(\theta) = \frac{ADJ}{HYP}, \quad \tan(\theta) = \frac{OPP}{ADJ}.}$$

GOAL: Given an integral of a particular form, we will use one of the TRIANGLES (I, II, III) along with SOHCAHTOA to CONVERT everything in the integral from "x" to "θ". One we do this, the integral will be computable. We then go back to "x" at the end! The is called the METHOD OF TRIGONOMETRIC SUBSTITUTION.

THREE TYPES of INTEGRALS

① $a^2 + x^2$ ② $\sqrt{a^2 - x^2}$ ③ $\sqrt{x^2 - a^2}$

REVIEW of INVERSE TRIG.

$\boxed{\text{arcsin}}$ (AKA $\sin^{-1}(x)$)

DOMAIN: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

* Arcsin "undoes" sine.

$$\arcsin(\sin(\theta)) = \theta$$

x	$\theta = \arcsin(x)$	
$x=0$	0	$\sin(?) = 0$
$x=1$	$\pi/2$	$\sin(?) = 1$

$\boxed{\text{arctan}}$

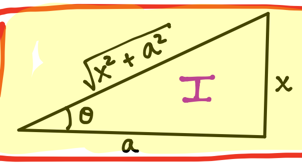
DOMAIN: $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

* Arctan "undoes" tangent.

$$\arctan(\tan\theta) = \theta$$

x	$\theta = \arctan(x)$
0	0
1	$\pi/4$.

PART 1. INTEGRALS CONTAINING "a^2+x^2" (A.K.A. TANGENT SUBSTITUTION)



form 2 x^2+4

Ex 1. $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\,dx$

Sol:

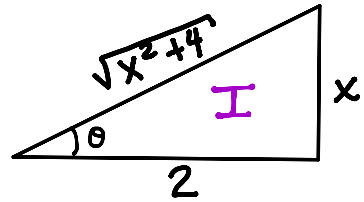
$$= \int \frac{1}{\left(\sqrt{x^2+4}\right)^3}\,dx$$

$$= \int \frac{2\sec^2\theta\, d\theta}{(2\sec\theta)^3} = \int \frac{2\sec^2\theta}{8\sec^3\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta + C = \boxed{\frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C}$$

go back to x. (use Δ)



officially we used $x = 2\tan\theta$ as our substitution

* $\tan\theta = \dfrac{x}{2}$ so $x = 2\tan\theta$, $\boxed{dx = 2\sec^2\theta\, d\theta}$

* I want $\sqrt{x^2+4}$ in terms of θ.

$\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$ so $\boxed{\sqrt{x^2+4} = 2\sec\theta}$

Ex 2. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+25}}$

form "x^2+25"

Sol:

$$= \int \frac{\cancel{5}\sec^{\cancel{2}}\theta\, d\theta}{25\tan^2\theta \cdot \cancel{5\sec\theta}}$$

$$= \frac{1}{25}\int \frac{\sec\theta\, d\theta}{\tan^2\theta}$$

Convert to sines & cosines.

$$= \frac{1}{25}\int \frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)^2}\,d\theta = \frac{1}{25}\int \frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}\,d\theta$$

$$= \frac{1}{25}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta = \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{25}\int \frac{\cancel{\cos\theta}}{w^2}\,\frac{dw}{\cancel{\cos\theta}}$$

$$= \frac{1}{25}\int w^{-2}\,dw = \frac{-1}{25}\cdot\frac{1}{w}$$

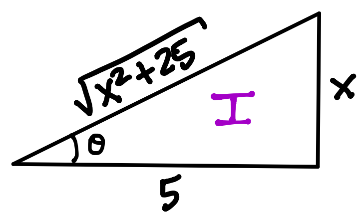
$w = \sin\theta$
$dw/d\theta = \cos\theta$
$d\theta = \dfrac{dw}{\cos\theta}$

$$= \frac{-1}{25}\cdot\frac{1}{\sin\theta} + C$$

$$= \frac{-1}{25}\csc\theta + C$$

use Δ.

$$= \boxed{-\frac{1}{25}\left(\frac{\sqrt{x^2+25}}{x}\right) + C}$$

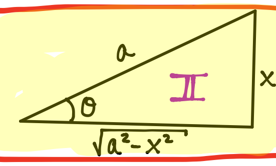


* $\tan\theta = \dfrac{x}{5} \Rightarrow x = 5\tan\theta$, $\boxed{dx = 5\sec^2\theta\, d\theta}$

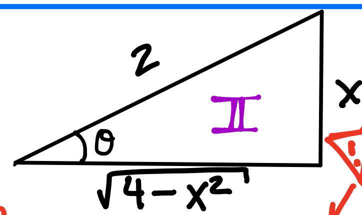
* $x^2 = (5\tan\theta)^2 = \boxed{25\tan^2\theta}$

* $\sec\theta = \dfrac{\sqrt{x^2+25}}{5} \Rightarrow \sqrt{x^2+25} = \boxed{5\sec\theta}$

PART 2. INTEGRALS CONTAINING "$\sqrt{a^2-x^2}$"
(A.K.A. SINE SUBSTITUTION)



Ex 3. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$ form "$4-x^2$"



official Subst.

$x=$

Sol:

$$= \int \frac{\cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}} = \int 1\, d\theta$$

$$= \theta + C$$

BACK TO x.

$$= \boxed{\arcsin\left(\frac{x}{2}\right) + C}$$

Solve for θ

$$\sin\theta = \frac{x}{2}$$

$$\arcsin(\sin\theta) = \arcsin\left(\frac{x}{2}\right)$$

$$\theta = \arcsin\left(\frac{x}{2}\right)$$

* $\underbrace{\sin\theta = \frac{x}{2}}$ $\quad (x = 2\sin\theta.)$

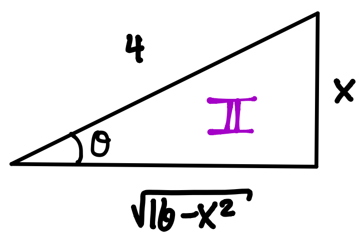
$dx = 2\cos\theta\, d\theta$

* $\cos\theta = \dfrac{\sqrt{4-x^2}}{2} \Rightarrow \sqrt{4-x^2} = 2\cos\theta$

Ex 4. $\displaystyle\int_{x=0}^{x=4} \frac{x^2}{\sqrt{16-x^2}}\,dx$ form "$\sqrt{16-x^2}$"

Sol:

CONVERT Bds:
* $\sin\theta = x/4$
$x=0$: $\sin\theta = 0$ so $\theta = 0$
$x=4$: $\sin\theta = 1$ $\theta = \pi/2$



$$\int \frac{16\sin^2\theta\ \cancel{4\cos\theta}\, d\theta}{\cancel{4\cos\theta}} = 16\int_{\theta=0}^{\theta=\pi/2} \sin^2\theta\, d\theta$$

½ ANGLE.

$$= 16\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= 8\int_0^{\pi/2} 1-\cos(2\theta)\, d\theta = 8\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/2}$$

$$8\left[\left(\frac{\pi}{2} - \cancel{\frac{\sin(\pi)}{2}}^{0}\right) - \left(0 - \cancel{\frac{\sin(0)}{2}}^{0}\right)\right] = \boxed{4\pi}$$

' $\sin\theta = \dfrac{x}{4}$ $\quad x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$x^2 = (4\sin\theta)^2 = 16\sin^2\theta$

$\cos\theta = \dfrac{\sqrt{16-x^2}}{4} \Rightarrow$

$\sqrt{16-x^2} = 4\cos\theta$

A NOTE ABOUT DOMAIN: $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

* HOW TO CONVERT

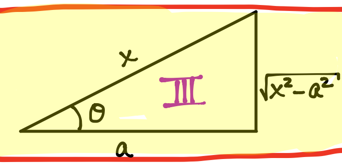
<u>EX4.</u> $\sin(2\theta)$ BACK TO x?

$$\sin(2\theta) = 2\sin\theta\cos\theta. = 2\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4}$$

DBL ∠ IDENTITY

FROM △

PART 3. INTEGRALS CONTAINING "$\sqrt{x^2-a^2}$"
(A.K.A. SECANT SUBSTITUTION)



Ex 5. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-2}}\,dx$

form "x^2-2"

Sol:

$$\int \frac{\cancel{\sqrt{2}\sec\theta}\,\cancel{\tan\theta}\,d\theta}{2\sec^{\cancel{2}}\theta\,\cancel{\sqrt{2}\tan\theta}}$$

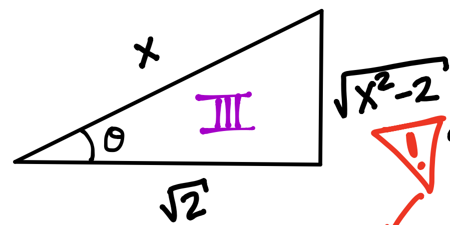
$$= \frac{1}{2}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta$$

$$= \frac{1}{2}\sin\theta + C$$

BACK to x.

$$= \frac{1}{2}\,\frac{\text{OPP}}{\text{HYP}} + C$$

$$= \boxed{\frac{1}{2}\,\frac{\sqrt{x^2-2}}{x} + C}$$



official Subst $\sqrt{2}\sec\theta = x$

* $\sec\theta = \dfrac{x}{\sqrt{2}} \Rightarrow x = \sqrt{2}\sec\theta$

  $dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$

* $x^2 = (\sqrt{2}\sec\theta)^2 = 2\sec^2\theta$.

* $\tan\theta = \dfrac{\sqrt{x^2-2}}{\sqrt{2}} \Rightarrow \sqrt{x^2-2} = \sqrt{2}\tan\theta$

A NOTE ABOUT DOMAIN: $0 \le \theta < \frac{\pi}{2}$

PART 4: SMARTER NOT HARDER

Try SIMPLE integration techniques FIRST!

Ex 6. $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$

$w = 9-x^2$

$\frac{dw}{dx} = -2x$

$dx = \frac{dw}{-2x}$

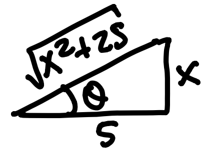
Sol:

$$\int \frac{\cancel{x}}{\sqrt{w}}\,\frac{dw}{-2\cancel{x}}$$

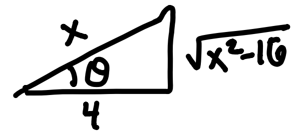
$$= -\frac{1}{2}\int w^{-1/2}\,dw = -\frac{1}{2}\,\frac{w^{1/2}}{\frac{1}{2}} + C = -\sqrt{w} + C = \boxed{-\sqrt{9-x^2} + C}$$

YOU MAY BE ASKED TO INDICATE THE APPROPRIATE TRIG SUBSTITUTION:
WHAT DOES THAT MEAN?

$\int \frac{1}{x^2+25}\,dx$ $\boxed{x = 5\tan\theta}$



$\int \frac{1}{\sqrt{x^2-16}}\,dx$ $\boxed{x = 4\sec\theta}$



$\int \frac{x^2}{\sqrt{16-x^2}}\,dx$ $\boxed{x = 4\sin\theta}$