CH 7.2: TRIG INTEGRALS

** We will be asked to compute some more involved TRIGONOMETRIC INTEGRALS. We will learn a technique of using TRIG IDENTITIES to rewrite the integrand, which will then enable us to compute the integral. First let's review some important facts about Trig functions.

PART 1. Some Trig IDENTITIES

PYTHAGOREAN IDENTITIES.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

RECIPROCAL IDENTITIES

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

DOUBLE ANGLE IDENTITIES

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta$$

HALF-ANGLE IDENTITIES

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

PRODUCT TO SUM FORMULAS

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$
$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

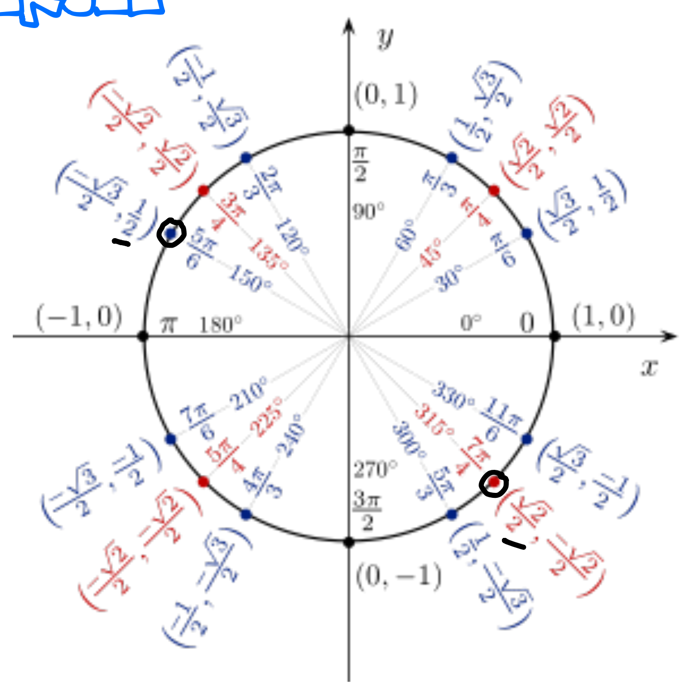
PART 2: THE UNIT CIRCLE

To evaluate $\sin(\theta)$ and $\cos(\theta)$ at a particular value of θ, the unit circle comes in handy!

- Find the correct angle θ on the unit circle.
- The point will have an ordered pair associated with it.
- The X VALUE represents $\cos(\theta)$
- The Y VALUE represents $\sin(\theta)$

$$\sin\left(\tfrac{5\pi}{6}\right) = \tfrac{1}{2}$$

$$\cos\left(\tfrac{7\pi}{4}\right) = \tfrac{\sqrt{2}}{2}$$



PART 3: TRIG INTEGRALS of Form $\int \cos^n(x)\,dx$, $\int \sin^n(x)\,dx$ OR $\int \sin^m(x)\cos^n(x)\,dx$ & sec(x), tan(x)

Ex 1. Evaluate the following definite and indefinite integrals:

A $\int \cos^2\theta\,d\theta = \int \left(\frac{1+\cos(2\theta)}{2}\right)d\theta$ ½ ANGLE

! EVEN POWER ON COSINE (NO ODD POWER)

Sol:

$$= \int \left(\frac{1}{2} + \frac{\cos(2\theta)}{2}\right)d\theta = \int \frac{1}{2}\,d\theta + \frac{1}{2}\int \cos(2\theta)\,d\theta$$

BASIC SUBST

$$= \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin(2\theta) + C$$

$$= \boxed{\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C}$$

! If This WERE "Definite" USE FUND THM of CALCULUS.

$$\int_0^{\pi/2} \cos^2\theta\,d\theta = \dots = \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi/2}$$ FUND THM.

$$\left(\frac{1}{2}\cdot\frac{\pi}{2} + \cancel{\frac{1}{4}\sin(\pi)}^{\,0}\right) - \left(0 + \cancel{\frac{1}{4}\sin(0)}^{\,0}\right) = \boxed{\frac{\pi}{4}}$$

B $\int \cos^3\theta\,d\theta = \int \cos\theta \cdot \cos^2\theta\,d\theta$ STRAGGLER

! ODD POWER ON COSINE

Sol:

PYTH. IDENT.

$$\int \cos\theta\,(1-\sin^2\theta)\,d\theta$$

$w = \sin\theta$

$dw/d\theta = \cos\theta.$

$d\theta = \dfrac{dw}{\cos\theta}.$

$$\int \cancel{\cos\theta}\,(1-w^2)\,\frac{dw}{\cancel{\cos\theta}}.$$

$$\int 1-w^2\,dw$$

$$= w - \frac{w^3}{3} + C = \boxed{\sin\theta - \frac{\sin^3\theta}{3} + C}$$

! $\int \cos^5\theta\,d\theta = \int \cos\theta \cdot \cos^4\theta\,d\theta = \int \cos\theta \cdot (\cos^2\theta)^2\,d\theta \ \cdots$ $\cos^2\theta \leftarrow 1-\sin^2\theta.$

C * $\int \sin^4(t)\,dt = \int \sin^2(t)\cdot \sin^2(t)\,dt$ ⚠ EVEN POWER ON SINE (NO ODD POWER)

½ ANGLE

sol:

$$\int\left(\frac{1-\cos(2t)}{2}\right)\left(\frac{1-\cos(2t)}{2}\right)dt = \frac{1}{4}\int 1-2\cos(2t)+\cos^2(2t)\,dt$$

$$= \frac{1}{4}\left[\int 1\,dt - 2\int \cos(2t)\,dt + \int \cos^2(2t)\,dt\right]$$

$$= \frac{1}{4}\left[t - \cancel{2}\,\frac{\sin(2t)}{\cancel{2}} + \int \frac{1+\cos(4t)}{2}\,dt\right]$$

$$= \frac{1}{4}\left[t - \sin(2t) + \frac{1}{2}\int 1\,dt + \frac{1}{2}\int \cos(4t)\,dt\right]$$

$$\boxed{= \frac{1}{4}\left[t - \sin(2t) + \frac{1}{2}t + \frac{1}{2}\cdot\frac{\sin(4t)}{4}\right] + C}$$

D $\int \sin(x)\cos^3(x)\,dx$

$w = \cos(x)$

$\frac{dw}{dx} = -\sin(x)$

$dx = \frac{dw}{-\sin(x)}$

sol:

$$\int \cancel{\sin(x)}\cdot w^3\cdot\frac{dw}{-\cancel{\sin(x)}}$$

$$-\int w^3\,dw = -\frac{w^4}{4} + C = \boxed{-\frac{\cos^4(x)}{4} + C}$$

SMARTER NOT HARDER!

EXPAND

E $\int 2(1+\sin\theta)^2\,d\theta = 2\int (1+2\sin\theta+\sin^2(\theta))\,d\theta.$

sol:

$$= 2\left[\int 1\,d\theta + 2\int \sin\theta\,d\theta + \int \sin^2(\theta)\,d\theta\right].$$

$$= 2\left[\theta - 2\cos\theta + \int \frac{1-\cos(2\theta)}{2}\,d\theta\right] \qquad \frac{1}{2}\int 1-\cos(2\theta)\,d\theta$$

$$= 2\left[\theta - 2\cos\theta + \int \frac{1}{2}\,d\theta - \frac{1}{2}\int \cos(2\theta)\,d\theta\right]$$

$$\boxed{= 2\left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{1}{2}\cdot\frac{\sin(2\theta)}{2}\right] + C}$$

E $\displaystyle\int \frac{\cos^3(\sqrt{x})}{\sqrt{x}}\,dx$ STRAGGLER

$w=\sqrt{x}=x^{1/2}$

$\frac{dw}{dx}=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$dx=2\sqrt{x}\,dw.$

$z=\sin(w)$

$\frac{dz}{dw}=\cos(w)$

$dw=\frac{dz}{\cos(w)}$

Sol:

$\displaystyle\int \frac{\cos^3(w)}{\cancel{\sqrt{x}}}2\cancel{\sqrt{x}}\,dw$

$=2\displaystyle\int \cos^3(w)\,dw.$

$=2\displaystyle\int \cos(w)\cos^2(w)\,dw$

STRAG.

$=2\displaystyle\int \cos(w)\,(1-\sin^2(w))\,dw.$

$2\displaystyle\int \cancel{\cos(w)}(1-z^2)\frac{dz}{\cancel{\cos(w)}}$

$=2\displaystyle\int 1-z^2\,dz$

$=2\left(z-\frac{z^3}{3}\right)+C$

$=\boxed{2\left(\sin(\sqrt{x})-\frac{\sin^3(\sqrt{x})}{3}\right)+C}$

F $\displaystyle\int \sin^5(x)\cos^2(x)\,dx = \int \sin(x)\cdot\underbrace{\sin^4(x)\cos^2(x)}_{\text{CONVERT TO }\cos(x)}dx$! ODD POWER ON SINE

straggler

Sol:

STRAG

$\displaystyle\int \sin(x)\cdot(\sin^2(x))^2\cos^2(x)\,dx = \boxed{\int \sin(x)\,(1-\cos^2(x))^2\cos^2(x)\,dx}$

$w=\cos(x)$

$dw/dx=-\sin(x)$

$dx=\frac{dw}{-\sin(x)}$

$\displaystyle\int \cancel{\sin(x)}\,(1-w^2)^2\cdot w^2\,\frac{dw}{-\cancel{\sin(x)}}$

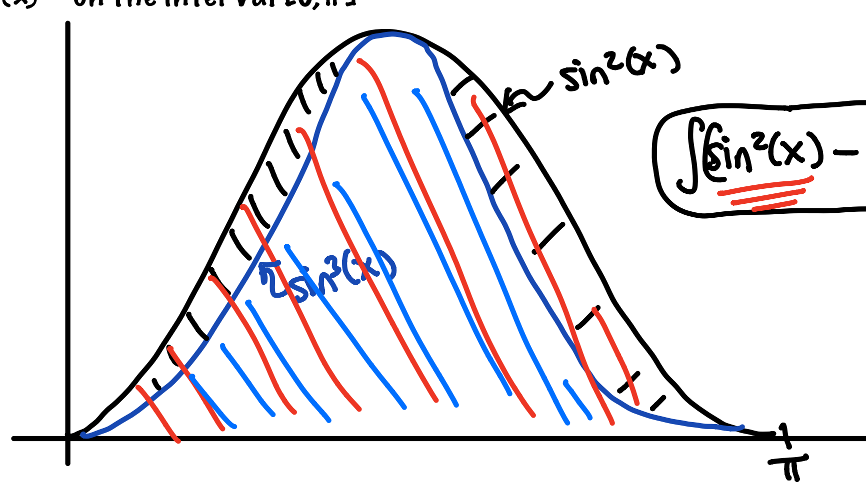
$-\displaystyle\int (1-w^2)^2\cdot w^2\,dw = -\int(1-2w^2+w^4)\cdot w^2\,dw$

$=-\displaystyle\int (w^2-2w^4+w^6)\,dw = -\left[\frac{w^3}{3}-\frac{2w^5}{5}+\frac{w^7}{7}\right]+C$

$=\boxed{-\left[\frac{\cos^3(x)}{3}-\frac{2\cos^5(x)}{5}+\frac{\cos^7(x)}{7}\right]+C}$

Ex 2. Set up an integral to express the area between the graphs $f(x)=\sin^2(x)$ and $f(x)=\sin^3(x)$ on the interval $[0,\pi]$

Sol:



$\boxed{\displaystyle\int (\sin^2(x)-\sin^3(x))\,dx}$

MORE **EXAMPLES**

① $\int \tan(x)\sec^4(x)\,dx = \int \underbrace{\sec(x)\tan(x)}_{\text{STRAGGLER}} \sec^3(x)\,dx$

if $w = \tan(x)$

STRAGGLER $\hookrightarrow \sec^2(x)$

if $w = \sec(x)$

STRAGGLER $\hookrightarrow \sec(x)\tan(x)$

$w = \sec(x)$

$\frac{dw}{dx} = \sec(x)\tan(x)$

$dx = \frac{dw}{\sec(x)\tan(x)}$

sol:

$$\int \cancel{\sec(x)\tan(x)} \cdot w^3 \cdot \frac{dw}{\cancel{\sec(x)\tan(x)}}$$

$$= \frac{w^4}{4} + C = \frac{\sec^4(x)}{4} + C$$

② $\int \tan(x)\cdot\sec^3(x)\,dx$

RECALL $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

sol: