

# CH 7.1 INTEGRATION BY PARTS "IBP"

**NOTE:** Integration by **SUBSTITUTION** undoes the **CHAIN RULE**. The next method that we will learn, **INTEGRATION by PARTS**, undoes the **PRODUCT RULE**.

## DERIVATION

of THE FORMULA

\*  $u$  &  $v$  ARE fns of  $x$

$$\int u dv = uv - \int v du$$

$$\frac{d}{dx}[u \cdot v] = uv' + v u' \quad \text{PROD. RULE}$$

$$\int \frac{d}{dx}[u \cdot v] dx = \int (uv' + v u') dx = \int uv' dx + \int v u' dx$$

$$v' = \frac{dv}{dx}$$
$$u' = \frac{du}{dx}$$

$$uv = \int u \cdot \frac{dv}{dx} dx + \int v \cdot \frac{du}{dx} dx$$

$$uv = \int u dv + \int v du \quad \rightarrow \quad \int u dv = uv - \int v du$$

**!** **INTEGRATION by PARTS** is useful when our integrand is a **product**. We choose " $u$ " and " $dv$ ", apply the formula, and hope that  $\int v du$  is simpler than  $\int u dv$

## PROCEDURE

**STEP 1:** Choose " $u$ " according to L.I.P.E.T.

L. I. P. E. T

logs.  $\uparrow$  L.

$\uparrow$  I. INVERSE TRIG

$\uparrow$  P. Polynomials ( $x, x^2, \text{etc.}$ )

$\leftarrow$  E. TRIG.

EXPONENTIAL fns.  $\nwarrow$

$$\int u dv = uv - \int v du$$

**STEP 2:** Let " $dv$ " be what's left!

**STEP 3:** Find " $du$ " by differentiating " $u$ " and find " $v$ " by integrating " $dv$ ". Then apply the formula!

PART 1: BASIC EXAMPLES

$$\int u dv = uv - \int v du$$

Ex 1 Compute the following indefinite integrals using INTEGRATION BY PARTS

A  $\int x e^x dx = \int u dv$   
sol:  $\Downarrow$  IBP

$u = x$   
 $dv = e^x dx$   
 WE CHOOSE.

$\frac{du}{dx} = 1 \Rightarrow du = dx$   
 $\int dv = \int e^x dx \Rightarrow v = e^x$   
 WE COMPUTE.

$$= uv - \int v du$$

$$= x e^x - \int e^x dx = x e^x - e^x + C$$

! TAKE A DERIVATIVE TO CHECK!

$$\frac{d}{dx} [x e^x - e^x + C] = x e^x + e^x - e^x + 0 = x e^x \quad \checkmark$$

B  $\int t \sin(t) dt = \int u dv$   
sol:  $\Downarrow$  IBP

$u = t$   
 $dv = \sin(t) dt$   
 WE CHOOSE.

$\frac{du}{dt} = 1 \Rightarrow du = dt$   
 $\int dv = \int \sin(t) dt \Rightarrow v = -\cos(t)$   
 WE COMPUTE.

$$= uv - \int v du$$

$$= -t \cos(t) - \int -\cos(t) dt$$

$$= -t \cos(t) + \int \cos(t) dt = -t \cos(t) + \sin(t) + C$$

$$\text{C} \int 30x^4 \ln(x) dx = 30 \int u dv$$

Sol:

$$\Downarrow \text{IBP} \\ = 30(uv - \int v du)$$

$$\begin{array}{l} \boxed{u = \ln(x)} \\ \boxed{dv = x^4 dx} \end{array}$$

WE CHOOSE.

$$\begin{array}{l} \text{DERIV.} \rightarrow \frac{du}{dx} = \frac{1}{x} \quad \boxed{du = \frac{dx}{x}} \\ \text{INT.} \rightarrow v = \int x^4 dx \quad \boxed{v = \frac{x^5}{5}} \end{array}$$

WE COMPUTE.

$$= 30 \left[ \frac{\ln(x) \cdot x^5}{5} - \int \frac{x^5}{5} \cdot \frac{dx}{x} \right]$$

$$\int \frac{x^4}{5} dx = \frac{1}{5} \int x^4 dx$$

$$= 30 \left[ \frac{\ln(x) \cdot x^5}{5} - \frac{x^5}{25} \right] + C$$

! WHAT if WE CHOOSE WRONG?  
 $\int u dv = uv - \int v du$   
 MORE COMPLICATED THAN

## PART 2: DEFINITE INTEGRALS

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Ex 2. Compute the following definite integrals using INTEGRATION by PARTS

$$\text{A} \int_1^2 \frac{\ln(x)}{x^2} dx = \int_1^2 \overset{u}{\ln(x)} \cdot \overset{dv}{\frac{1}{x^2} dx}$$

Sol:

$$\Downarrow \text{IBP} \\ = uv - \int v du$$

$$\begin{array}{l} \boxed{u = \ln(x)} \\ \boxed{dv = \frac{1}{x^2} dx} \end{array}$$

WE CHOOSE.

$$\begin{array}{l} \text{DERIV.} \rightarrow \frac{du}{dx} = \frac{1}{x} \quad \boxed{du = \frac{dx}{x}} \\ \text{INT.} \rightarrow v = \int x^{-2} dx \quad \boxed{v = \frac{-1}{x}} \end{array}$$

WE COMPUTE.

$$= \ln(x) \cdot \left(\frac{-1}{x}\right) - \int \frac{-1}{x} \cdot \frac{dx}{x} = -\frac{\ln(x)}{x} + \int x^{-2} dx$$

$$= \left( -\frac{\ln(x)}{x} - \frac{1}{x} \right) \Big|_1^2$$

$$= \left( -\frac{\ln(2)}{2} - \frac{1}{2} \right) - \left( -\frac{\ln(1)}{1} - \frac{1}{1} \right)$$

### PART 3. IT WORKS!

For some integrals, it may not look like integration by parts will apply, but it still works!

Ex 3. Compute the following integrals using **INTEGRATION by PARTS**

A  $\int (\ln(4x)) \cdot 1 \cdot dx = \int u \cdot dv$

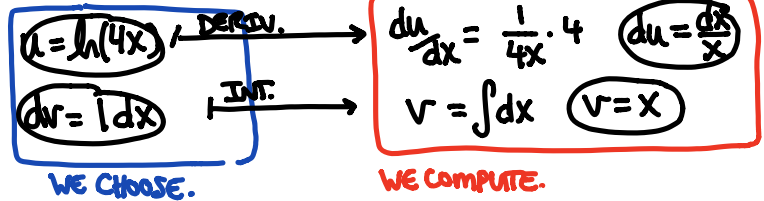
Sol:

IBP

$$= uv - \int v du$$

$$= \ln(4x) \cdot x - \int x \cdot \frac{dx}{x}$$

$$= x \ln(4x) - \int dx = x \ln(4x) - x + C$$



B  $\int \arctan(2x) \cdot 1 \cdot dx$

Sol:

$$= \int u \cdot dv$$

IBP

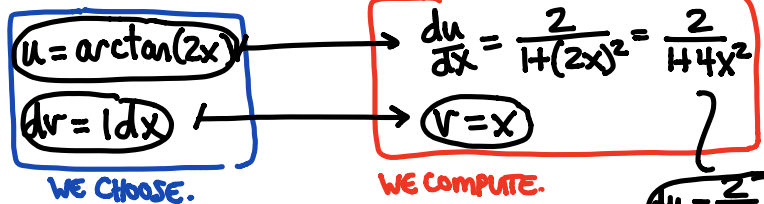
$$= uv - \int v du$$

$$= x \arctan(2x) - \int \frac{2x \cdot dx}{1+4x^2}$$

METHOD of SUBSTITUTION.

$$w = 1+4x^2$$

$$\frac{d}{dx}[\arctan(u)] = \frac{u'}{1+u^2}$$



$$du = \frac{2}{1+4x^2} dx$$

PART 4: MORE **COMPLEX** EXAMPLES

Ex 4. [DOUBLE BY PARTS]  $\int t^2 \cos(t) dt$

IBP #1

Sol:

$$= \int u dv$$

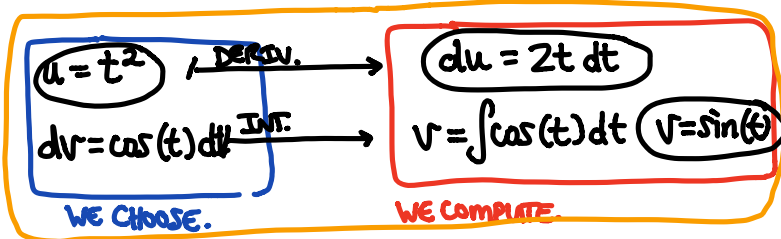
↙ IBP

$$= uv - \int v du$$

$$= t^2 \sin(t) - \int 2t \sin(t) dt.$$

$$= t^2 \sin(t) - 2 \int t \sin(t) dt = t^2 \sin(t) - 2 [t \cos(t) + \sin(t)] + C$$

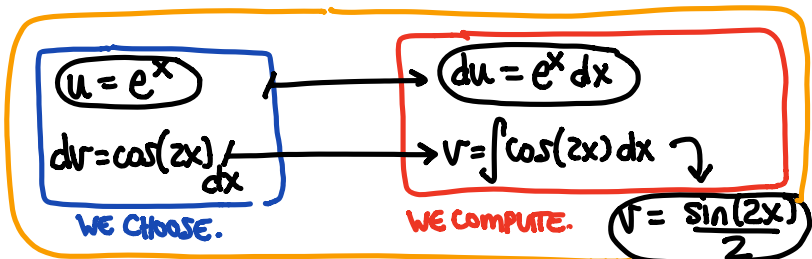
Ex 1B.  
IBP #2.



### Ex 5. [THE GENIUS MOVE]

Sol:  $\int e^x \cos(2x) dx$   
 $= \int u dv$   
 $\Downarrow$  IBP

IBP #1



$$\int e^x \cos(2x) dx = uv - \int v du = \frac{e^x \sin(2x)}{2} - \int \frac{e^x \sin(2x)}{2} dx$$

$$\int e^x \cos(2x) dx = \frac{e^x \sin(2x)}{2} - \frac{1}{2} \int e^x \sin(2x) dx$$

$$\int e^x \cos(2x) dx = \frac{e^x \sin(2x)}{2} - \frac{1}{2} \left[ -\frac{e^x \cos(2x)}{2} + \frac{1}{2} \int e^x \cos(2x) dx \right]$$

$$\int e^x \cos(2x) dx = \frac{e^x \sin(2x)}{2} + \frac{e^x \cos(2x)}{4} - \frac{1}{4} \int e^x \cos(2x) dx$$

$$\int e^x \cos(2x) dx = \frac{4}{5} \left[ \frac{e^x \sin(2x)}{2} + \frac{e^x \cos(2x)}{4} \right] + C$$

IBP #2

$$\int e^x \sin(2x) = \dots = -\frac{e^x \cos(2x)}{2} + \frac{1}{2} \int e^x \cos(2x) dx$$

$$\left\{ \begin{array}{l} \square \\ +\frac{1}{4}\square \end{array} \right. = \triangle - \frac{1}{4}\cancel{\square} + \frac{1}{4}\square$$

$$\text{Ang. } \left\{ \begin{array}{l} \frac{5}{4} \square = \triangle \\ \square = \frac{4}{5} \triangle \end{array} \right.$$