CH 7.1 INTEGRATION BY PARTS "IBP"

NOTE: Integration by SUBSTITUTION undoes the CHAIN RULE. The next method that we will learn, INTEGRATION by PARTS, undoes the PRODUCT RULE.

DERIVATION of THE FORMULA

$$\boxed{\int u\,dv = uv - \int v\,du}$$

* u & v ARE fn's of x

$$\frac{d}{dx}[u\cdot v] = uv' + vu' \qquad \text{PROD. RULE}$$

$v' = \frac{dv}{dx}$

$u' = du/dx.$

$$\int \frac{d}{dx}[u\cdot v]\,dx = \int (uv' + vu')\,dx = \int uv'\,dx + \int vu'\,dx$$

$$uv = \int u\cdot\frac{dv}{\cancel{dx}}\cancel{dx} + \int v\cdot\frac{du}{\cancel{dx}}\cancel{dx}$$

$$uv = \int u\,dv + \int v\,du \rightarrow \boxed{\int u\,dv = uv - \int v\,du}$$

! INTEGRATION by PARTS is useful when our integrand is a product. We choose "u" and "dv", apply the formula, and hope that $\int v\,du$ is simpler than $\int u\,dv$

PROCEDURE

$$\boxed{\int u\,dv = uv - \int v\,du}$$

STEP 1: Choose "u" according to L.I.P.E.T.

L. I. P. E. T

- L: logs.
- I: INVERSE TRIG
- P: POLYNOMIALS $(x, x^2, \text{etc.})$
- E: EXPONENTIAL fn's
- T: TRIG.

STEP 2: Let "dv" be what's left!

STEP 3: Find "du" by differentiating "u" and find "v" by integrating "dv". Then apply the formula!

PART 1: BASIC EXAMPLES

$$\int u\,dv = uv - \int v\,du$$

Ex 1 Compute the following indefinite integrals using INTEGRATION by PARTS

A $\int \underbrace{x}_{u}\,\underbrace{e^x dx}_{dv} = \int u\,dv$

sol:

$\Downarrow$ IBP

$= uv - \int v\,du$

$= xe^x - \int e^x dx = \boxed{xe^x - e^x + C}$

WE CHOOSE.		WE COMPUTE.
$u = x$	DERIV. $\rightarrow$	$\frac{du}{dx} = 1$ $\quad du = dx$
$dv = e^x dx$	INT. $\rightarrow$	$\int dv = \int e^x dx$. $\quad v = e^x$

TAKE A Derivative to CHECK!

$$\frac{d}{dx}\left[xe^x - e^x + C\right] = xe^x + \cancel{e^x} - \cancel{e^x} + 0$$

$$= \boxed{xe^x}$$

B $\int \underbrace{t}_{u}\,\underbrace{\sin(t)\,dt}_{dv} = \int u\,dv$

sol:

$\Downarrow$ IBP

$= uv - \int v\,du$

$= -t\cos(t) - \int -\cos(t)\,dt$

$= -t\cos(t) + \int \cos(t)\,dt = \boxed{-t\cos(t) + \sin(t) + C}$

WE CHOOSE.		WE COMPUTE.
$u = t$	DERIV. $\rightarrow$	$\frac{du}{dt} = 1$ $\quad du = dt$
$dv = \sin(t)\,dt$	INT. $\rightarrow$	$\int dv = \int \sin(t)\,dt$ $\quad v = -\cos(t)$

C $\int 30x^4 \ln(x)\,dx = 30\int u\,dv$

Sol:

$\Downarrow$ IBP

$= 30\left(uv - \int v\,du\right)$

$u = \ln(x) \xrightarrow{\text{DERIV.}} \frac{du}{dx} = \frac{1}{x} \quad du = \frac{dx}{x}$

$dv = x^4 dx \xrightarrow{\text{INT.}} v = \int x^4 dx \quad v = \frac{x^5}{5}$

WE CHOOSE. WE COMPUTE.

$= 30\left[\ln(x)\cdot\frac{x^5}{5} - \underline{\int \frac{x^5}{5}\cdot\frac{dx}{x}}\right]$

$\int \frac{x^4}{5}\,dx = \frac{1}{5}\int x^4\,dx$

$= 30\left[\ln(x)\cdot\frac{x^5}{5} - \frac{x^5}{25}\right] + C$

WHAT if WE CHOOSE WRONG?

$\int u\,dv = uv - \int v\,du$

MORE COMPLICATED THAN

PART 2: DEFINITE INTEGRALS

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Ex 2. Compute the following definite integrals using INTEGRATION by PARTS

A $\int_1^2 \frac{\ln(x)}{x^2}\,dx = \int_1^2 \ln(x)\cdot\frac{1}{x^2}\,dx$ (u, dv)

Sol:

$\Downarrow$ IBP

$= uv - \int v\,du$

$u = \ln(x) \xrightarrow{\text{DERIV.}} \frac{du}{dx} = \frac{1}{x} \quad du = \frac{dx}{x}$

$dv = \frac{1}{x^2}dx \xrightarrow{\text{INT.}} v = \int x^{-2}dx \quad v = \frac{-1}{x}$

WE CHOOSE. WE COMPUTE.

$\frac{x^{-1}}{-1}$

$= \ln(x)\cdot\left(\frac{-1}{x}\right) - \int \frac{-1}{x}\cdot\frac{dx}{x} = -\frac{\ln(x)}{x} + \underline{\int x^{-2}dx}$

$= \left(-\frac{\ln(x)}{x} - \frac{1}{x}\right)\Big|_1^2$

$x=2 \qquad x=1$

$= \left(-\frac{\ln(2)}{2} - \frac{1}{2}\right) - \left(-\frac{\ln(1)}{1} - \frac{1}{1}\right)$

PART 3. IT WORKS!

For some integrals, it may not look like integration by parts will apply, but it still works!

Ex 3. Compute the following integrals using INTEGRATION by PARTS

$$\frac{d}{dx}[\ln(u)] = \frac{u'}{u}$$

A $\int \underbrace{\ln(4x)}_{u}\,\underbrace{1\cdot dx}_{dv} = \int u\,dv$

Sol:

$\Downarrow$ IBP

WE CHOOSE:

$u = \ln(4x) \xrightarrow{\text{DERIV.}} \frac{du}{dx} = \frac{1}{4x}\cdot 4 \qquad du = \frac{dx}{x}$

$dv = 1dx \xrightarrow{\text{INT.}} v = \int dx \qquad v = x$

WE COMPUTE.

$$= uv - \int v\,du$$

$$= \ln(4x)\cdot x - \int \cancel{x}\cdot\frac{dx}{\cancel{x}}$$

$$= x\ln(4x) - \int dx = \boxed{x\ln(4x) - x + C}$$

$$\frac{d}{dx}[\arctan(u)] = \frac{u'}{1+u^2}$$

B $\int \arctan(2x)\cdot 1\,dx$

Sol: $= \int u\,dv$

$\Downarrow$ IBP

WE CHOOSE:

$u = \arctan(2x) \longrightarrow \frac{du}{dx} = \frac{2}{1+(2x)^2} = \frac{2}{1+4x^2} \qquad du = \frac{2}{1+4x^2}dx$

$dv = 1dx \longrightarrow v = x$

WE COMPUTE.

$$= uv - \int v\,du$$

$$= x\arctan(2x) - \int \frac{2x\,dx}{1+4x^2}$$

METHOD of SUBSTITUTION.

$$w = 1+4x^2$$

PART 4: MORE COMPLEX EXAMPLES

Ex 4. [DOUBLE BY PARTS] $\int t^2\cos(t)\,dt$

Sol:

$$= \int u\,dv$$

$\Downarrow$ IBP

IBP #1

WE CHOOSE.		WE COMPUTE.
$u = t^2$	$\xrightarrow{\text{DERIV.}}$	$du = 2t\,dt$
$dv = \cos(t)\,dt$	$\xrightarrow{\text{INT.}}$	$v = \int \cos(t)\,dt$ $v = \sin(t)$

$$= uv - \int v\,du$$

$$= t^2\sin(t) - \int 2t\sin(t)\,dt.$$

$$= t^2\sin(t) - 2\int t\sin(t)\,dt = t^2\sin(t) - 2\left[t\cos(t) + \sin(t)\right] + C$$

Ex 1B.
IBP #2.

Ex 5. [THE GENIUS MOVE]

Sol: $\int e^x \cos(2x)\,dx$

$= \int u\,dv$

$\Downarrow$ IBP

IBP #1

$u = e^x \longrightarrow du = e^x\,dx$

$dv = \cos(2x)\,dx \longrightarrow v = \int \cos(2x)\,dx$

$v = \dfrac{\sin(2x)}{2}$

WE CHOOSE. WE COMPUTE.

$$\int e^x \cos(2x)\,dx = uv - \int v\,du = \frac{e^x \sin(2x)}{2} - \int \frac{e^x \sin(2x)}{2}\,dx$$

$$\int e^x \cos(2x)\,dx = \frac{e^x \sin(2x)}{2} - \frac{1}{2}\int e^x \sin(2x)\,dx$$

$$\int e^x \cos(2x)\,dx = \frac{e^x \sin(2x)}{2} - \frac{1}{2}\left[-\frac{e^x \cos(2x)}{2} + \frac{1}{2}\int e^x \cos(2x)\,dx\right]$$

$$\int e^x \cos(2x)\,dx = \frac{e^x \sin(2x)}{2} + \frac{e^x \cos(2x)}{4} - \frac{1}{4}\int e^x \cos(2x)\,dx$$

$$\boxed{\int e^x \cos(2x)\,dx = \frac{4}{5}\left[\frac{e^x \sin(2x)}{2} + \frac{e^x \cos(2x)}{4}\right] + C}$$

IBP #2

$$\int e^x \sin(2x) = \dots = -\frac{e^x \cos(2x)}{2} + \frac{1}{2}\int e^x \cos(2x)\,dx$$

$$\square = \Delta - \cancel{\tfrac{1}{4}\square}$$

$$+\tfrac{1}{4}\square \qquad \cancel{+\tfrac{1}{4}\square}$$

Alg.

$$\begin{cases} \frac{5}{4}\square = \triangle \\ \square = \frac{4}{5}\triangle \end{cases}$$