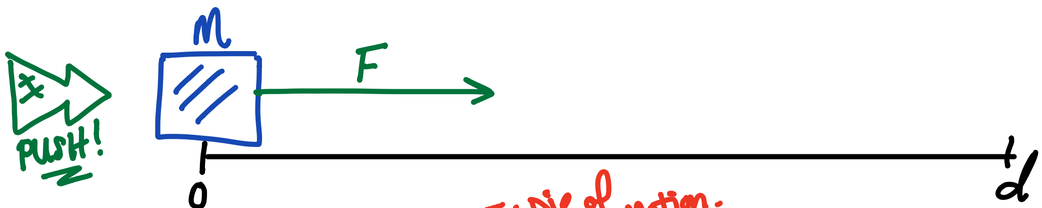
Ch 6.4 WORK: An Application of Integration

Part 1: The BASICS of Work.

A constant Force:



IN DIR of motion.

DEFN 1: [Work by Constant Force]

$$W = F \cdot d$$

! UNITS:

	d	F	W
SI	m	N	J
ENGLISH	ft	lb	ft-lb.

Newton
Joule.

1 ft = 0.305 m
1 ft-lb = 1.36 J
1 lb = 4.45 N

DEFN 2: [Gravitational Force] ON EARTH

$$F_g = m \cdot g$$

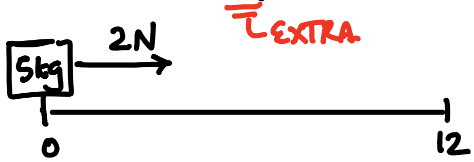
g = Acceler due to gravity

$g = 9.81\ m/s^2$ SI
$g = 32\ ft/s^2$ ENGLISH.

Ex 1: Calculate the WORK done in the following situations:

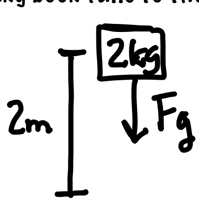
A A force of 2 Newtons moves a 5kg box 12 meters.

EXTRA



$W = 2 \cdot 12 = \boxed{24\ J}$

B A 2kg book falls to the ground from a height of 2 meters.

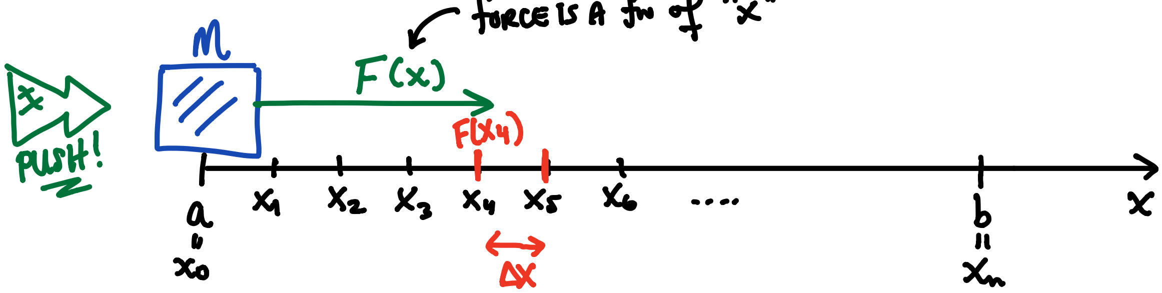


$W = F_g \, d = mg\, d = \boxed{4g\ J}$

QUESTION: WHAT IF FORCE IS NOT CONSTANT?

ANSWER: WE NEED CALCULUS! (INTEGRALS).

force is a fn of "x"



Work on this interval $\approx F(x_4)\cdot\Delta x$

Work to move from x_j to x_{j+1} $\approx F(x_j)\cdot\Delta x$

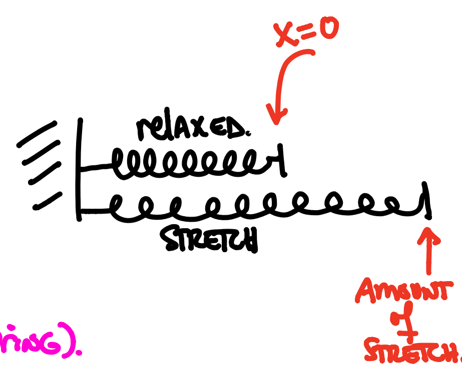
ADD THEM ALL.

Total Work $\approx \displaystyle\sum_{j=0}^{n-1} F(x_j)\Delta x$

LET $\Delta x \to 0$

$$\boxed{\text{Work} = \int_a^b F(x)\,dx}$$

PART 2: THE SPRING PROBLEM



DEFN 3: [HOOKE'S LAW].

Force to maintain a spring @ a distance of "x" from relaxed state is $F(x) = kx$

SPRING CONSTANT (Diff for every Spring).

Ex 1 A force of 8 lbs is required to hold a spring stretched 2 inches from its natural length. How much work is done in stretching ~~its natural length~~ to 4 inches beyond its natural length.

Sol. Hooke's Law (find k)

$F = kx$ ⚠ 2"

$8\text{ lb} = k\cdot(\tfrac{1}{6}\text{ ft})$

$\boxed{k = 48\ \text{lb}/\text{ft}}$

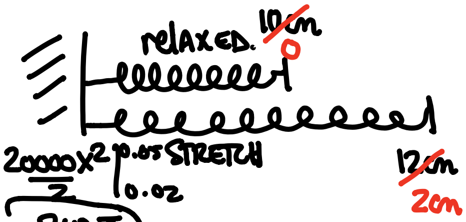
$W = \displaystyle\int_{x=a}^{x=b} F(x)\,dx = \int_0^{1/3} 48x\,dx$ ⚠ 4"

Hooke.

$= 24x^2\Big|_0^{1/3}$

$= \boxed{\tfrac{24}{9}\text{ ft-lbs}}$

Ex 2 Suppose that 4 J or work is needed to stretch a spring from its natural length of 10 cm to a length of 12 cm. SI



A How much work is needed to stretch the spring from 12 cm to 15 cm?

Sol:

$$W = \int_a^b \underbrace{F(x)}_{\text{Hooke}}\,dx = \int_{0.02}^{0.05} kx\,dx = \int_{0.02}^{0.05} 20000x\,dx = \frac{20000x^2}{2}\Big|_{0.02}^{0.05} = \boxed{210J}$$

STRETCH

100cm = 1m

Find k

$$W = 4J = \int_0^{0.02} kx\,dx = \frac{kx^2}{2}\Big|_0^{0.02} = \frac{k(0.02)^2}{2} = 4J \qquad \boxed{k = 20000\ \frac{N}{m}}$$

B How far beyond its natural length will a force of 20 Newtons keep the spring stretched.

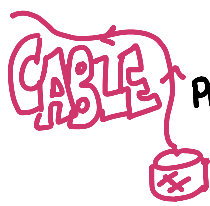
Sol:

Hooke's Law $\quad F = kx = 20000x$

20N

Find x.

$$x = \frac{20}{20000} = \boxed{\frac{1}{1000}\ m = x}$$

Part 3: The  Problem

*We can apply similar techniques to another application!

Ex 3. A cable with density 2 lb/ft is used to lift 100 lb bundle of shingles from the base of a building to the roof. Treating the shingles as a concentrated point mass, determine the work needed to hoist the shingles up to the roof that is 20 feet tall.

ENGLISH.

A What is the weight of a slice of the cable with length Δy?

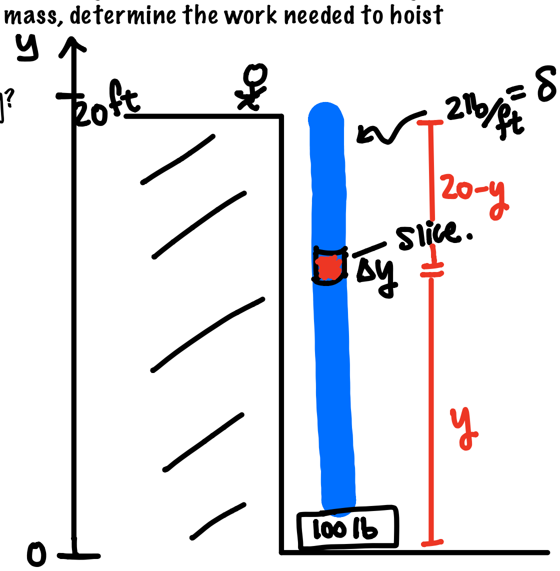
*Treat shingles + cable separate.

Sol:

delta density. $\rightarrow \delta = 2\text{lb}/\text{ft} = \dfrac{\text{WEIGHT}}{\text{LENGTH}}$

$$\text{WEIGHT} = \delta \cdot \text{LENGTH}.$$

$$\boxed{\text{Weight of slice} = \delta \cdot \Delta y}$$



B Compute the work done to lift the cable by itself to the roof?

Sol:

$$\text{Work for the slice} = \text{Force} \cdot \text{Distance}$$

$$= \left(\text{Weight of slice}\right) \cdot \text{Distance}$$

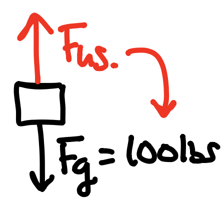
Work for 1 slice.

$$= (\delta \cdot \Delta y)(20 - y) = \boxed{\delta(20-y)\Delta y}$$

Add up y = 0 to y = 20

Total work for cable.

$$\boxed{W = \int_0^{20} \underset{\overset{\shortparallel}{2}}{\delta}(20-y)\,dy} = 400 \text{ ft-lbs}$$



C Compute the work done in lifting the shingles to the top of the roof, and add this to the answer from part B.

Sol:

Work for shingles

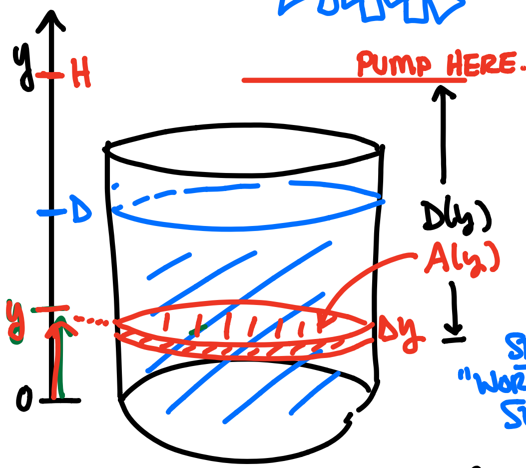
$$W = F \cdot d = \text{Weight} \cdot d$$

$$= 100\text{lbs} \cdot 20\text{ft} = 2000 \text{ ft-lbs}$$

$$\text{Total work} = 400 + 2000 = 2400 \text{ ft-lbs}$$

PART 4: THE TANK PROBLEM.

*WE CAN COMPUTE WORK NECESSARY TO PUMP FLUIDS FROM A TANK!



$A(y)$ = Cross Sectional Area of slice.

$D(y)$ = Distance to pump slice.

δ = fluid density

SI

$\delta = \underset{\text{DENSITY}}{\text{MASS}} = \frac{kg}{m^3} = \frac{\text{MASS}}{\text{VOL.}}$

$\text{MASS} = \delta \cdot \text{VOL}$

STEP 1 "WORK FOR SLICE"

$$
\begin{aligned}
W_{slice} &= F \cdot d \\
&= \text{Weight} \cdot d \\
&= \text{Mass}_{\text{Slice}} \cdot g \cdot d \\
&= \delta \cdot g \cdot d \cdot \text{Vol}_{\text{of Slice}} \\
&= \delta g d \cdot A(y)\,\Delta y \\
&= \boxed{\delta g D(y) A(y)\,\Delta y}
\end{aligned}
$$

STEP 2 "TOTAL WORK"

$$\boxed{W = \int_0^D \delta g\, D(y) A(y)\, dy}$$

ENGLISH

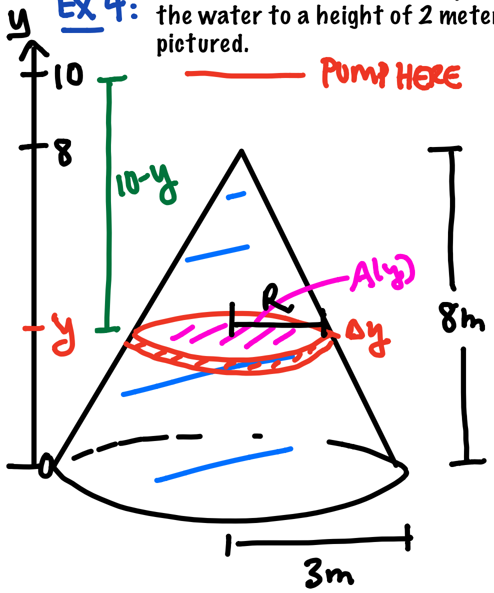
$\delta = \underset{\text{DENSITY}}{\text{WEIGHT}} = \frac{lb}{ft^3} = \frac{\text{WEIGHT}}{\text{VOL}}$

$\text{WEIGHT} = \delta \cdot \text{VOL}$

$$
\begin{aligned}
W_{slice} &= F\, d \\
&= \text{weight} \cdot d \\
&= \delta \cdot \text{Vol}_{\text{of Slice}} \cdot d \\
&= \delta \cdot A(y) \underset{D(y)}{\underbrace{d}}\, \Delta y \\
&= \boxed{\delta A(y) D(y)\, \Delta y}
\end{aligned}
$$

$$\boxed{W = \int_0^D \delta D(y) A(y)\, dy}$$

Ex 4: A tank if full of water. Set up an integral that represents the work required to pump the water to a height of 2 meters above the top of the tank. The tank is shaped as pictured.



$\delta = 1000 \text{ kg/m}^3$

S.I.

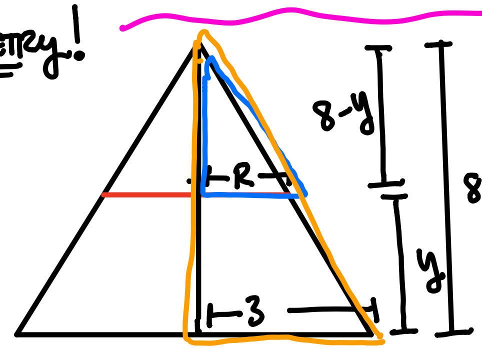
$$W = \int_0^8 \delta \cdot g \cdot D(y)\, A(y)\, dy$$

$D(y) = 10 - y$

$A(y) = \pi R^2 = \pi\left(\frac{3}{8}(8-y)\right)^2$

$$\boxed{W = \int_0^8 \delta g \pi \left(\frac{3}{8}(8-y)\right)^2 \cdot (10-y) \cdot dy} \text{ J}$$

* GEOMETRY!



$A(y) = \pi R^2$ R in terms of y

$$\frac{R}{3} = \frac{8-y}{8}$$

$$\boxed{R = \frac{3}{8}(8-y)}$$