CH 11.9 POWER SERIES REPRESENTATION of FUNCTIONS

GOAL: We are going to learn about an important application of power series. Using what we know about geometric series, we can represent certain functions (on particular intervals) as power series!

PART 1: FINDING POWER SERIES REPRESENTATIONS

GEO SERIES: $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$ $\begin{cases} \text{CONVERGES TO } \frac{a}{1-r} \text{ if } |r|<1 \\ \text{DIVERGES if } |r| \geq 1 \end{cases}$

** Replace r with the variable x and let a=1 and we get the following:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \overset{*}{=} \frac{1}{1-x} \quad \text{PROVIDED } |x|<1 \quad (\text{i.e } x \in (-1,1))$$

POWER SERIES REPRESENTATION of $f(x) = \frac{1}{1-x}$

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

! if $x \in (-1,1)$. "in"

WHY DO WE CARE?:
- * $f(x) = \frac{1}{1-x}$ YUK!
- * $1 + x + x^2 + \dots$ YAY!

** We can use this to find POWER SERIES REPRESENTATIONS and corresponding intervals of convergence for other functions that have a "similar" form.

Ex 1. Find the POWER SERIES REPRESENTATION of each of the following functions and indicate the interval of convergence.

$$\boxed{\sum_{n=0}^{\infty} (\square)^n = \frac{1}{1-\square} \quad \text{if } -1 < \square < 1}$$

A $f(x) = \frac{1}{1-2x} = \frac{1}{1-\boxed{2x}}$

sol:

$$= \sum_{n=0}^{\infty} (2x)^n = \boxed{\sum_{n=0}^{\infty} 2^n x^n}$$

I.o.C $-1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

$$f(x) = \sum_{n=0}^{\infty} 2^n x^n \text{ when } x \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

B $f(x)=\frac{1}{1+x^2} = \frac{1}{1-\boxed{-x^2}}$

sol:

$$= \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n \cdot x^{2n}$$

$$\boxed{\sum_{n=0}^{\infty}(\square)^n = \frac{1}{1-\square} \quad \text{if } -1<\square<1 \ \left(\overset{\text{OR}}{|\square|<1}\right)}$$

I.o.C $\quad -1 < -x^2 < 1$

$1 > x^2 > -1$

$\Rightarrow \quad |x^2| < 1 \quad \Rightarrow \quad |x| < 1$ I.o.C $(-1,1)$

C $f(x)=\frac{4}{x+5} = \frac{4}{5+x} = 4\cdot\left(\frac{1}{5+x}\right)$

sol:

$$= 4\left(\frac{\frac{1}{5}}{\frac{5}{5}+\frac{x}{5}}\right) = \frac{4}{5}\left(\frac{1}{1-\boxed{\left(-\frac{x}{5}\right)}}\right)$$

$$\boxed{\sum_{n=0}^{\infty}(\square)^n = \frac{1}{1-\square} \quad \text{if } -1<\square<1 \ \left(\overset{\text{OR}}{|\square|<1}\right)}$$

$$= \frac{4}{5}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n = \frac{4}{5}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{5^n}$$

$$\left(-\frac{x}{5}\right)^n = \left(\frac{-1\cdot x}{5}\right)^n$$

I.o.C $\quad \left|-\frac{x}{5}\right| < 1 \quad$ so $\quad |x|<5$

I.o.C. $(-5,5)$

D $f(x)=\frac{6}{x^2-2x-8}$ HINT: PARTIAL FRACTIONS.

sol:

E $f(x) = \frac{x}{1-x^5} = x \cdot \left(\frac{1}{1-\boxed{x^5}}\right)$

Sol:

$$= x \sum_{n=0}^{\infty} (x^5)^n$$

$$= x \sum_{n=0}^{\infty} x^{5n} = \boxed{\sum_{n=0}^{\infty} x^{5n+1}}$$

I.o.C $\quad -1 < x^5 < 1$

$-1 < x < 1$ $\quad \boxed{(-1,1)}$

$$\boxed{\sum (\square)^n = \frac{1}{1-\square} \quad \text{if } |\square| < 1}$$

PART 2: CALCULUS w/ POWER SERIES

RECALL:

$$\frac{d}{dx}[f_1(x) \pm f_2(x)] = \frac{d}{dx}[f_1(x)] \pm \frac{d}{dx}[f_2(x)]$$

$$\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$$

TERM-BY-TERM DIFFERENTIATION/ INTEGRATION.

* This property of derivatives and Integrals still holds for an infinite sum!

THM: [DERIVATIVES/INTEGRALS of POWER SERIES]

If the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ has radius of convergence $R>0$, then the function:

$$f(x) = \sum C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$$

is differentiable (and hence continuous) on the interval $(a-R, a+R)$ and:

- The DERIVATIVE is $\frac{d}{dx}[f(x)] = \frac{d}{dx}[C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots]$

  * TERM By TERM.

- The INTEGRAL is $\int f(x)\,dx = \int (C_0 + C_1(x-a) + C_2(x-a)^2 \cdots)\,dx.$

NOTE: if POWER SERIES of $f(x)$ HAS R.o.C "R"

THEN $\frac{d}{dx}[f(x)]$ & $\int f(x)\,dx$ HAVE R.o.C "R"

MAY NOT PRESERVE I.o.C @ ENDPTS.

A SHORTCUT

W/ SUMMATION NOTATION

Ex. $\frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]$

METHOD 1: $\frac{d}{d\cancel{x}}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[x^n]$

$= \sum_{n=\cancel{0}1}^{\infty} n x^{n-1}$

METHOD 2:

$\frac{d}{dx}\left[1+x+x^2+x^3+x^4+\dots\right] = 0+1+2x+3x^2+\dots$

Ex 2: Consider the function $f(x)=\frac{1}{(1-x)^2}$

[A] Use differentiation to find a POWER SERIES REPRESENTATION for f(x).
What is the radius of convergence of this new power series?

sol:

$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]$

$\underbrace{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}_{\text{TAKE DER BOTH SIDES.}}$

R.o.C = 1

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$

So $f(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ R.o.C. = 1

[B] Use part [A] to find a POWER SERIES REPRESENTATION for $g(x)=\frac{1}{(1-x)^3}$
What is the new radius of convergence?

sol: der of BOTH SIDES of [A]

$\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{d}{d\cancel{x}}\left[\sum_{n=1}^{\infty} n \cancel{x}^{n-1}\right]$

$(1-x)^{-2}$

$-2(1-x)^{-3}\cdot -1$

$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$

$g(x) = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}$ R.o.C = 1.

C Use part [B] to find a POWER SERIES REPRESENTATION for $h(x)=\frac{x}{(1-x)^3}$
What is the new radius of convergence?

sol:

$$h(x) = x \cdot g(x) = x \cdot \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)\,x^{n-2}$$

$$= \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-1}$$

R.o.C $= 1$

Ex 3. Evaluate the integral as a power series and indicate the radius of convergence.

$$\int \frac{x}{1-x^5}\,dx$$

sol: