CH 11.8: POWER SERIES

THE MAIN IDEA:

What numbers can we input into this series to have the result be a CONVERGENT series?

$$\sum_{n=0}^{\infty} \frac{(\boxed{}-1)^n}{2^n}$$

$x=1$ $\quad \sum_{n=0}^{\infty} 0$ CONVERGES

$x=2$ $\quad \sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ CONV.

$x=3$ $\quad \sum \frac{2^n}{2^n} = \sum 1$ DIV.

$x=-2$ $\quad \sum \frac{(-3)^n}{2^n} = \sum \left(\frac{-3}{2}\right)^n$ DIV.

* Typically, the green box is replaced by a variable "x" and we create what is called a POWER SERIES.

$$\sum \frac{(x-1)^n}{2^n} = \sum \underbrace{\frac{1}{2^n}}_{C_n}(x-1)^n \quad a=1.$$

DEFN: [POWER SERIES]

A POWER SERIES centered at "a" is a series with the form:

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a)^1 + C_2(x-a)^2 + C_3(x-a)^3 + \ldots$$

Where "x" is a variable and the C_n's are constants that we refer to as the coefficients of the series.

NOTE:

(1) The C_n's are functions of "n" so for each value of n, they become a constant. We call them constants because they are constant with respect to the variable x. Today we will consider C_n's that have the form:

$$\frac{1}{2^n},\ \frac{1}{n!},\ n!,\ \frac{(-1)^n}{n+1},\ \frac{2^{n+1}}{\sqrt{n}},\ \text{etc.}$$

(2) A power series can be thought of as a POLYNOMIAL of INFINITE DEGREE. For example:

$C_n = 1/n \quad a=0.$

$$\text{Power Series} = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

(3) Each different value of x yields a different series. For some values of x this series may converge and for others it may diverge... That's where we come in!

GOAL: Determine the values of "x" for which $\sum_{n=0}^{\infty} C_n(x-a)^n$ will converge.

Ex 1. Find all values of x for which the following power series converges:

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} \qquad a=1$$

sol:

sol: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$ GEO SERIES. $r = \frac{x-1}{2}$

CONVERGES if $|r| < 1$ $\qquad \left|\frac{x-1}{2}\right| < 1$ ALGEBRA.

$$\frac{|x-1|}{2} < 1$$

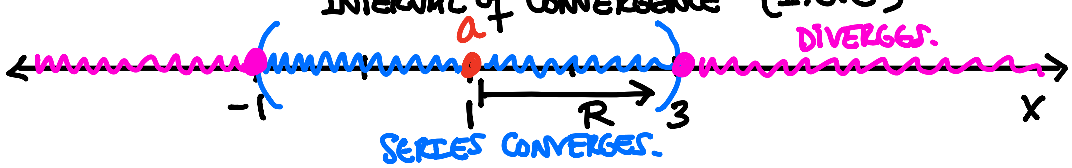
$$|x-1| < 2$$

$$-2 < x-1 < 2$$

$$-1 < x < 3$$

SERIES CONVERGES for $-1 < x < 3$ $\quad (-1, 3)$

"INTERVAL of CONVERGENCE" (I.O.C)



R = "RADIUS of CONVERGENCE" (R.O.C)

$$\boxed{\begin{array}{l} \text{I.O.C} = (-1, 3) \\ \text{R.O.C} = 2 \end{array}}$$

There are three different possibilities for a given power series: $\sum_{n=0}^{\infty} c_n (x-a)^n$

*Proof w/ RATIO TEST.

- The series converges for all values of x. In this case:

RADIUS of CONVERGENCE
$R.o.C = \infty$

INTERVAL OF CONVERGENCE
$I.o.C = (-\infty, \infty)$

- The series converges only when x=a. In this case:

RADIUS of CONVERGENCE
$R.o.C = 0$

INTERVAL OF CONVERGENCE
$I.o.C = \{a\}$

- There exists some value R>0 (called the radius of convergence) such that the series converges for $|x-a| < R$. In this case:

$-R < x-a < R$
$a-R < x < R+a$

RADIUS of CONVERGENCE
$R.o.C = R$

INTERVAL OF CONVERGENCE
$[$ or $(a-R, a+R)$ or $]$

! WE MUST CHECK ENDPTS INDIVIDUALLY for CONVERGENCE

THE PROCEDURE

USE RATIO TEST.

RECALL

THM: [THE RATIO TEST]

$\sum a_n$. $\quad \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L$

$L < 1$ L=0 is o.k.	$L > 1$ L=∞ is o.k.	$L = 1$
ABS. CONVERGENT	DIVERGENT	WE KNOW NOTHING!

Ex 2. Find the RADIUS of CONVERGENCE and INTERVAL of CONVERGENCE for each of the following power series:

A $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $a=0.$ $a_n = \frac{x^n}{n!}$ $|a_n| = \frac{|x^n|}{n!} = \frac{|x|^n}{n!}$ $|a_{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$

Sol: RATIO TEST

$x \neq 0$

$$\lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty}\left(\frac{|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n}\right) = \lim_{n\to\infty}\left(|x|\cdot\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}\right)$$

$$= \lim_{n\to\infty}\left(|x|\cdot\frac{1}{n+1}\right) = |x|\cdot\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0 = L$$

$L<1$ (Regardless of x) so CONVERGES for ALL x.

$\mathbb{R}$

R.o.C $= \infty$
I.o.C $= (-\infty, \infty)$

B $\sum_{n=1}^{\infty} n!\,x^n$ $a=0$ $a_n = n!\,x^n$ $|a_n| = n!\,|x|^n$ $|a_{n+1}| = (n+1)!\,|x|^{n+1}$

Sol: $x \neq 0$

$$\lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty}\left(\frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n}\right) = |x|\lim_{n\to\infty}((n+1)) = \infty = L$$

$L>1$ (REGARDLESS of x) so DIVERGES for ALL x EXCEPT $x=0$

R.o.C $= 0$ I.o.C $= \{0\}$

C $\sum_{n=1}^{\infty} \frac{(-1)^n\cdot(x-1)^n}{n}$ $a=1$ $a_n = \frac{(-1)^n(x-1)^n}{n}$ $|a_n| = \frac{|x-1|^n}{n}$ $|a_{n+1}| = \frac{|x-1|^{n+1}}{n+1}$

Sol:

$x \neq 1$

$$\lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty}\left(\frac{|x-1|^{n+1}}{n+1}\cdot\frac{n}{|x-1|^n}\right) = |x-1|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= |x-1|\cdot 1 = L$$

CONVERGE if $L<1$. i.e if $|x-1| < 1$

R.o.C. $R=1$

$|x-1|<1$
$-1 < x-1 < 1$
$0 < x < 2$

I.o.C $=]a-R, a+R[$
$= [(0, 2]] = (0, 2]$

CHECK ENDPTS
* INPUT VALUE TO SERIES & SEE IF IT CONV. OR DIVERGES

$\boxed{x=0}$ $\sum \frac{(-1)^n(-1)^n}{n} = \sum \frac{(-1)^{2n}}{n} = \sum \frac{1}{n}$ DIVERGES By P-TEST

$\boxed{x=2}$ $\sum \frac{(-1)^n\cdot 1}{n}$ CONVERGE By A.S.T.

$\boxed{D}$ $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{3^n \cdot n^2}$ $a=-2$ $|a_n| = \frac{|x+2|^n}{3^n \cdot n^2}$ $|a_{n+1}| = \frac{|x+2|^{n+1}}{3^{n+1} \cdot (n+1)^2}$

Sol: $x \neq -2$ $\lim_{n\to\infty} \left(\frac{|a_{n+1}|}{|a_n|} \right) = \lim_{n\to\infty} \left(\frac{|x+2|^{n+1}}{3^{n+1} \cdot (n+1)^2} \cdot \frac{3^n \cdot n^2}{|x+2|^n} \right)$

$= \frac{|x+2|}{3} \lim_{n\to\infty} \left(\frac{n^2}{(n+1)^2} \right) = \frac{|x+2|}{3} = L$

CONVERGES if $\frac{|x+2|}{3} < 1$ so $|x+2| < 3$ R.O.C $R=3$

CHECK ENDPTS ! $I.o.C = [-5, 1]$ $I.o.C = a-R, a+R$

$= [-5, 1]$

$\boxed{x=-5}$

$\sum \frac{(-1)^n (-3)^n}{3^n \cdot n^2} = \sum \frac{(-1)^n (-1 \cdot 3)^n}{3^n \cdot n^2} = \sum \frac{(-1)^n \cdot (-1)^n \cdot \cancel{3^n}}{\cancel{3^n} \cdot n^2} = \sum \frac{1}{n^2}$ CONVERGE P-TEST

$\boxed{x=1}$ $\sum \frac{(-1)^n \cancel{3^n}}{\cancel{3^n} \cdot n^2} = \sum \frac{(-1)^n}{n^2}$ CONVERGES A.S.T.

Ex 3. (i) Find the RADIUS of CONVERGENCE and INTERVAL of CONVERGENCE for each of the following power series: $\sum_{n=1}^{\infty} \frac{2^{n+1}}{\sqrt{n}} \cdot x^n$

Sol:

(ii) Knowing what we know from part (i), what can we tell about the following series:

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{\sqrt{n}} \cdot x^{2n}$$

sol:

Ex 4. Suppose that the **RADIUS of CONVERGENCE** of the power series $\sum_{n=0}^{\infty} C_n x^n$ is R. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} C_n x^{3n}$

$a=0$

sol:

$\sum C_n x^n$ CONVERGES for $-R < x < R$

$a-R < x < a+R$

So $\sum C_n x^{3n} = \sum C_n (x^3)^n$ CONVERGES $-R < x^3 < R$

$\sqrt[3]{-R} < x < \sqrt[3]{R}$

$\sqrt[3]{R}$ = R.o.C

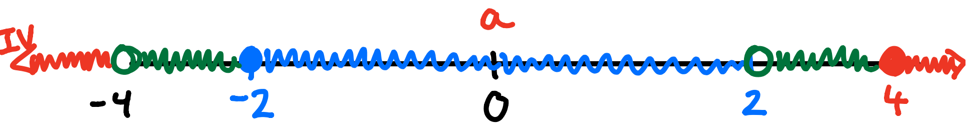
Ex 5: Suppose that $\sum_{n=0}^{\infty} c_n x^n$ ~ $a=0$. converges when x=-2 and diverges when x=4. What can be said about the CONVERGENCE/DIVERGENCE of the following series?

*UNKNOWN
*CONVERGE
*DIVERGE

Sol:

$\boxed{A}$ $\sum c_n(-1)^n$ x CONV.

$\boxed{B}$ $\sum c_n(5)^n$ DIV



$\boxed{C}$ $\sum c_n\cdot(2)^n$ UNK

$\boxed{D}$ $\sum c_n(1)^n$ CONV.