CH 11.6 ABSOLUTE CONVERGENCE and RATIO TEST

GOAL: We will discuss a property of convergence that some series possess and we will learn our final convergence test of this unit!

PART 1: ABSOLUTE and CONDITIONAL CONVERGENCE

* Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$ ← Some terms may be negative.

we consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + \dots$$

DEFN: A series $\sum a_n$ is:

- **ABSOLUTELY CONVERGENT** if: $\sum |a_n|$ converges. ($\sum a_n$ converges for free!)
- **CONDITIONALLY CONVERGENT** if: $\sum a_n$ converges but $\sum |a_n|$ diverges
- **DIVERGENT** if: $\sum a_n$ diverges.

Note that if the series has all positive terms then $|a_n| = a_n$ so absolute convergence is the same as convergence in this case.

Ex 1. Determine if the following series are ABSOLUTELY CONVERGENT, CONDITIONALLY CONVERGENT, or DIVERGENT.

A $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (a_n) ~~AC~~, CC, ~~D~~

Sol: $\sum a_n$

$\sum \frac{(-1)^{n-1}}{n}$

AST $b_n = 1/n$

(i) $\lim_{n\to\infty} b_n = 0$ ✓

(ii) $b_{n+1} \leq b_n$ ✓

$\frac{1}{n+1} \leq \frac{1}{n}$

Converges by AST

$\sum |a_n|$

$\sum \frac{1}{n}$

Diverges by P-TEST.

So $\sum \frac{(-1)^{n-1}}{n}$ is conditionally conv.

B $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ AC, ~~CC~~, ~~D~~

Sol: $\sum a_n$

No work needed.

$\sum |a_n| = \sum \frac{1}{n^2}$

Converges by P-TES.

So $\sum \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent

$\boxed{C}$ $\sum_{n=1}^{\infty}(-1)^n\dfrac{n}{\sqrt{n^3+1}}$ ~~AC~~ CC ~~D~~

sol: $\sum a_n = \sum(-1)^n\dfrac{n}{\sqrt{n^3+1}}$

AST SEE Ex E of 11.5

CONVERGES by AST

$\sum|a_n| = \sum\dfrac{n}{\sqrt{n^3+1}}$ BEHAVES LIKE $\sum\dfrac{1}{n^{1/2}}$ WHICH DIVERGES By P-TEST.

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^3+1}}\cdot\frac{n^{1/2}}{1}\right)=\lim_{n\to\infty}\left(\frac{n^{3/2}}{\sqrt{n^3+1}}\right)=1$$

C

So $\sum\dfrac{n}{\sqrt{n^3+1}}$ DIVERGES by Lim Comp TEST.

So $\sum(-1)^n\dfrac{n}{\sqrt{n^3+1}}$ is CONDITIONALLY CONVERGENT

$\boxed{D}$ $\sum_{n=1}^{\infty}\dfrac{\cos(2n)}{2^n}$

sol:

* There is an important relationship between the convergence of $\sum a_n$ and $\sum|a_n|$

THM: If a series $\sum a_n$ is **ABSOLUTELY CONVERGENT** then it is CONVERGENT.

(I.e. If $\sum|a_n|$ converges then $\sum a_n$ converges too!)

So if $\sum a_n$ DIVERGES THEN $\sum|a_n|$ DIVERGES.

PART 2: THE RATIO TEST

GEO & AST.

* Currently, we only have ~~one~~ Two test for convergence that applies to series with positive and negative terms. We will now learn a ~~second~~ Third one, called the RATIO TEST! This test is very useful in determining if a series is absolutely convergent.

THM: [THE RATIO TEST]

Given the series $\sum a_n$. if: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$

$L < 1$ ($L=0$ too)	$L > 1$ ($L=\infty$ too)	$L = 1$
$\sum a_n$ is ABSOLUTELY CONVERGENT	$\sum a_n$ is DIVERGENT.	WE KNOW <u>NOTHING</u>

Ex 2. Use the **RATIO TEST** to determine if each series is **ABSOLUTELY CONVERGENT**, **CONDITIONALLY CONVERGENT**, or **DIVERGENT**

! $\frac{x^6}{x^5} = x \qquad \frac{x^a}{x^b} = x^{a-b}$

A $\sum_{n=1}^{\infty} \frac{6^n}{n^2} \qquad |a_n| = \frac{6^n}{n^2} \qquad |a_{n+1}| = \frac{6^{n+1}}{(n+1)^2}$

Sol:

$$\lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty}\left(\frac{6^{n+1}}{(n+1)^2}\cdot\frac{n^2}{6^n}\right) = \lim_{n\to\infty}\left(\frac{6^{n+1}\cdot n^2}{6^n\cdot(n+1)^2}\right)$$

$$6\lim_{n\to\infty}\left(\frac{n^2}{(n+1)^2}\right) = 6\cdot 1 = 6 = L$$

$L > 1 \implies$ $\boxed{\sum \frac{6^n}{n^2} \text{ DIVERGES by RATIO TEST}}$

B $\sum_{n=1}^{\infty} \frac{(-2)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n\cdot 2^n}{n\cdot 3^n} \qquad |a_n| = \frac{2^n}{n\cdot 3^n} \qquad |a_{n+1}| = \frac{2^{n+1}}{(n+1)\cdot 3^{n+1}}$

Sol:

$$\lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty}\left(\frac{2^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n\cdot 3^n}{2^n}\right) = \frac{2}{3}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= \frac{2}{3}\cdot 1 = L$$

$L < 1$ so $\boxed{\sum \frac{(-2)^n}{n\cdot 3^n} \text{ ABSOLUTELY CONVERGES By RATIO TEST}}$

The RATIO TEST works great with series involving factorials! This means that you will need to be comfortable working with factorials.

" ! "

$$4! = 4\cdot 3\cdot 2\cdot 1 = \boxed{24}$$

$$n! = n\cdot(n-1)(n-2)\cdots 2\cdot 1$$

$$(n+2)! = (n+2)(n+1)(n)\cdots 2\cdot 1$$

$$(2n)! = (2n)(2n-1)(2n-2)\cdots 2\cdot 1$$

$$0! = 1$$

EXPAND

$$\frac{8!}{4!} \neq 2!$$

$$(2n)! \neq 2\cdot n!$$

C $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$ $\quad |a_n| = \dfrac{n!}{2^n} \quad |a_{n+1}| = \dfrac{(n+1)!}{2^{n+1}}$

sol:

$$\lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty}\left(\frac{(n+1)!}{2^{n+1}}\cdot\frac{2^n}{n!}\right) = \lim_{n\to\infty}\left(\frac{2^n}{2^{n+1}}\cdot\frac{(n+1)!}{n!}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{(n+1)\cancel{(n)}\cancel{(n-1)}\cdots\cancel{2\cdot 1}}{\cancel{n}\cancel{(n-1)}\cdots\cancel{2\cdot 1}}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{(n+1)\cdot\cancel{n!}}{\cancel{n!}}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}(n+1) = \infty = L > 1$$

So $\sum \dfrac{n!}{2^n}$ DIVERGES BY RATIO TEST

D $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n)!}$

sol:

E $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ $\quad |a_n| = \dfrac{(2n)!}{(n!)^2}$ $\quad |a_{n+1}| = \dfrac{(2(n+1))!}{((n+1)!)^2} = \dfrac{(2n+2)!}{((n+1)!)^2}$ ⚠

Sol:

$$\lim_{n\to\infty}\left(\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty}\left(\frac{(2n+2)!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)!\cdot n!\; n!}{(2n)!\,(n+1)!\,(n+1)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)\cdot(2n+1)\cdot\cancel{(2n)!}\;\cancel{n!}\;\cancel{n!}}{\cancel{(2n)!}\;(n+1)\cancel{n!}\;(n+1)\cancel{n!}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\right) = \lim_{n\to\infty}\left(\frac{4n^2+6n+2}{n^2+2n+1}\right)$$

$$= 4 = L > 1$$

So $\sum \dfrac{(2n)!}{(n!)^2}$ DIVERGES By RATIO TEST

⚠ The RATIO TEST is inconclusive for P-Series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$