CH 11.5 ALTERNATING SERIES +/−

NOTE: So far, the tests for convergence that we have learned have only applied to series with positive terms. In this section, we will consider series that have terms which are not always positive. Particularly, we will look at ALTERNATING SERIES where terms alternate being positive/negative.

PART 1: THE GENERAL FORM of AN ALTERNATING SERIES

* $(-1)^n$ OR $(-1)^{n+1}$ OR SIMILAR.

Ex 1. Write out the first 5 terms of the following ALTERNATING SERIES:

[A] $\sum_{n=1}^{\infty} \underbrace{(-1)^n \cdot \frac{1}{n}}_{a_n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

Sol: $b_n = 1/n$

[B] $\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} \cdot 2^n}_{a_n} = 2 - 4 + 8 - 16 + 32 - \cdots$

Sol: $b_n = 2^n$

DEFN: [ALTERNATING SERIES]

A SERIES w/ TERMS THAT ALTERNATE.

$$\sum \underbrace{(-1)^n \cdot b_n}_{a_n} \quad \text{OR} \quad \sum \underbrace{(-1)^{n+1} b_n}_{a_n}$$

(OR SIMILAR).

w/ $b_n \geq 0$.

The convergence of an alternating series will depend only on b_n

$b_n = |a_n|$.

** To determine if an alternating series CONVERGES, we can apply the ALTERNATING SERIES TEST

THM: [ALTERNATING SERIES TEST]: A.S.T.

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - \cdots \quad \text{w/ } b_n > 0$$

Satisfies both:

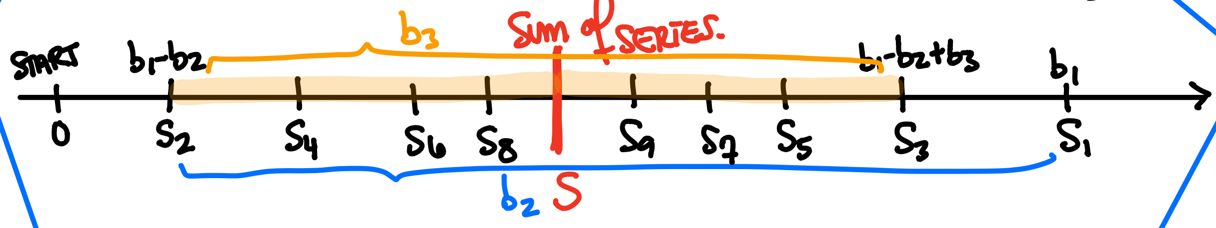
b_n's DECREASE.

(i) $\lim_{n \to \infty} b_n = 0$ AND (ii) $b_{n+1} \leq b_n$ for ALL n

Then the series is CONVERGENT. Otherwise, the test is INCONCLUSIVE

PROOF IDEA: PARTIAL SUMS. $S_N = \sum_{n=1}^{N} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots + b_N$

$S_1 = b_1$, $S_2 = b_1 - b_2$, $S_3 = b_1 - b_2 + b_3$, $S_4 = b_1 - b_2 + b_3 - b_4$



The **DIVERGENCE TEST** still applies to alternating series. So if the terms of the series DO NOT tend toward zero, then the series **DIVERGES**

if (i) FAILS THEN SERIES DIVERGES by DIV. TEST.

Ex 2: Determine if the following series **CONVERGE** or **DIVERGE**. You must provide proper justification.

$\boxed{A}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (ALTERNATING HARMONIC)

Sol: A.S.T. $b_n = 1/n$

CHECK HYPOTHESES

(i) $\lim_{n\to\infty} b_n = 0$ ✓

(ii) CHECK $b_{n+1} \le b_n$ for ALL n

$$\frac{1}{n+1} \overset{?}{\le} \frac{1}{n} \quad ✓$$

So by A.S.T $\sum \frac{(-1)^n}{n}$ CONVERGES

Why does the harmonic series **DIVERGE** and the alternating harmonic series **CONVERGE**?

$$\sum 1/n = 1 + 1/2 + 1/3 + \dots$$

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

* PRESENCE of NEGATIVES "TAMES" GROWTH!

$\boxed{B}$ $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{4n+3}$ a_n A.S.T

Sol: $b_n = \frac{n}{4n+3}$ CHECK HYPOTH (i) $\lim_{n\to\infty}\left(\frac{n}{4n+3}\right) = \frac{1}{4} \neq 0.$ X

A.S.T inconclusive.

By DIVERGENCE TEST $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4n+3}$ DIVERGES $\lim_{n\to\infty}\underbrace{\left(\frac{(-1)^n \cdot n}{4n+3}\right)}_{a_n} \neq 0.$

$\boxed{C}$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^3+1}$ A.S.T.

Sol: $b_n = \frac{n}{n^3+1}$ CHECK HYPOTH

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{n}{n^3+1}\right) = 0$ ✓

(ii) CHECK $b_{n+1} \leq b_n$ ✓

* b_n is DECREASING.

$\frac{n+1}{(n+1)^3+1} \overset{?}{\leq} \frac{n}{n^3+1}$

$f(x) = \frac{x}{x^3+1}$ SHOW $f'(x) < 0.$

$f'(x) = \frac{-2x^3+1}{(x^3+1)^2}$ ⊕

$f'(x) < 0$ WHEN $-2x^3+1 < 0$

$1 < 2x^3$

$\sqrt[3]{\frac{1}{2}} < x$ ✓

By A.S.T $\sum \frac{(-1)^{n-1} n}{n^3+1}$ CONVERGES

D $\sum_{n=1}^{\infty}(-1)^n e^{2/n}$! Visually check limit first.

sol: $\lim_{n\to\infty}(e^{2/n}) = e^0 = 1 \neq 0.$ so $\lim_{n\to\infty}((-1)^n e^{2/n}) \neq 0$

so $\boxed{\sum(-1)^n e^{2/n} \text{ DIVERGES BY DIV. TEST}}$

E $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+1}}$ A.S.T.

sol: $b_n = \frac{n}{\sqrt{n^3+1}}$

(i) $\lim_{n\to\infty} b_n = 0$ ✓

(ii) b_n's DECREASE. ✓

$f(x) = \frac{x}{\sqrt{x^3+1}}$ SHOW $f'(x) < 0$. ! EVENTUALLY!

$$f'(x) = \frac{\sqrt{x^3+1} - \frac{3/2\,x^3}{\sqrt{x^3+1}}}{(x^3+1)}\left(\frac{\sqrt{x^3+1}}{\sqrt{x^3+1}}\right) = \frac{x^3+1-3/2\,x^3}{(x^3+1)^{3/2}}$$ +

$f'(x) < 0 \iff x^3+1-3/2\,x^3 < 0$

$-\frac{1}{2}x^3 < -1 \qquad x^3 > 2 \qquad x > \sqrt[3]{2}$

$\boxed{\text{By A.S.T } \sum(-1)^n \cdot \frac{n}{\sqrt{n^3+1}} \text{ CONVERGE}}$

PART 2: ESTIMATING SUMS of ALTERNATING SERIES.

NOTE: As we have seen, a partial sum of any convergent series can be used to estimate the sum of the series. When using an approximation, we always have error and an associated REMAINDER. For alternating series, we can find a BOUND for the size of the remainder when an nth partial sum is used.

$S_N = N^{th}$ PARTIAL

$S_N = \sum_{n=1}^{N}(-1)^n b_n$

REMAINDER: $\boxed{R_N = S - S_N}$

EXACT APPROXIMATE

THM: [ALTERNATING SERIES ESTIMATION]

Suppose $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ is a convergent alternating series (i.e. satisfies conditions of A.S.T.) with $b_n > 0$, then the following holds:

$$\text{SUM} = S = \sum_{n=1}^{\infty}(-1)^n \cdot b_n \qquad |R_N| = |S - \underbrace{S_N}_{N^{th}\text{ PARTIAL SUM.}}| \leq \overbrace{b_{N+1}}^{\text{BOUND}}$$

Ex 3. How many terms do we need to add in order to approximate the SUM of the following convergent series with an error less than 0.0001? Then find this approximation.

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n} \qquad b_n = 1/n \qquad b_{n+1} = \frac{1}{n+1}$$

Sol:

$\boxed{\text{EXACT SUM}}$

$$S = \sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n}$$

$\boxed{\text{APPROXIMATE}}$

$$S_N = \sum_{n=1}^{N}(-1)^n \cdot \frac{1}{n}$$

$$|R_N| = |S - S_N| < \underbrace{0.0001}_{\text{SET ACCURACY}}$$

$$|R_N| = |S - S_N| \leq b_{N+1} < 0.0001$$

$$\frac{1}{N+1} < 0.0001 \qquad \text{SOLVE FOR } N$$

$$N > 9999 \qquad N = 10{,}000$$

Ex 4. Let S_N be the Nth partial sum of the convergent series $S = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n+1}$.

$\boxed{A}$ Give a bound on the error $|S - S_{100}|$ as a decimal rounded to three decimal places.

$\boxed{B}$ Use the bound on the remainder to find an N such that $|S - S_N| \leq 0.001$

Sol: $\boxed{A}$

$$|S - S_{100}| \leq b_{N+1} = b_{101} = \frac{1}{3(101)+1} = \frac{1}{304} \qquad b_n = \frac{1}{3n+1}$$

$$N = 100$$

$\boxed{B}$

$$|S - S_N| \leq b_{N+1} \leq 0.001$$

$$\frac{1}{3(N+1)+1} \leq 0.001 \qquad \text{SOLVE FOR } N.$$

$$N \geq 332 \qquad N = 332$$