CH 11.4: COMPARISON TESTS

GOAL: In chapter 11.3 we compared a series to a known improper integral to determine if the series converged or diverged. Now we will compare a given series to another series!

TWO COMPARISON TESTS

THE COMPARISON TEST | THE LIMIT COMPARISON TEST

PART 1 THE COMPARISON TEST

MAIN IDEA

- If the terms of a given series are GREATER THAN (or equal to) the terms of a series that is known to be DIVERGENT, then the given series must DIVERGE
- If the terms of a given series are LESS THAN (or equal to) the terms of a series that is known to be CONVERGENT, then the given series must CONVERGE

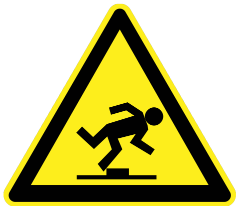
THM: [THE COMPARISON TEST]

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is CONVERGENT and $a_n \le b_n$ for all n, then $\sum a_n$ is also CONVERGENT.
- If $\sum b_n$ is DIVERGENT and $a_n \geq b_n$ for all n, then $\sum a_n$ is also DIVERGENT.

PROCEDURE: To determine if $\sum a_n$ converges or diverges by comparison test:

1. DETERMINE a COMPARISON SERIES: look only at the higher power (or most dominant term) in the numerator and denominator to determine your comparison series.
2. DOES COMPARISON SERIES CONVERGE or DIVERGE? Use known results about geometric series and p-series.
3. SHOW the APPROPRIATE INEQUALITY: Use algebra. If comparison series diverges then you must show $\geq$. If it converges, you must show $\leq$. If you are unable to do so, comparison test may not work!



Some common series to use for comparison:

- **GEOMETRIC SERIES:** $\sum_{n=0}^{\infty} ar^n$ CONVERGE if $|r|<1$, DIVERGE if $|r| \geq 1$
- **P-SERIES:** $C\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGE if $p>1$, DIVERGE if $p \leq 1$

NOTE: Although the comparison test requires us to check inequalities "for all n", we only need to verify that it works "for all n past a certain point". This is because the convergence of a series is unaffected by a finite number of terms so we can effectively just start our series at a later point.

Ex 1. Use the COMPARISON TEST to determine if the following series converge or diverge. Be sure to provide proper justification.

[A] $\sum_{n=1}^{\infty} \frac{4}{n^2+3n+5}$

Sol: THE SERIES $\sum \frac{4}{n^2+3n+5}$ (a_n) BEHAVES LIKE $\sum \frac{4}{n^2}$ (b_n). WHICH CONVERGES BY P- TEST. WE SHOW: $a_n \leq b_n$. NOTICE:

$$a_n = \frac{4}{n^2+3n+5} \leq \frac{4}{n^2} = b_n$$

THEREFORE THE SERIES $\sum \frac{4}{n^2+3n+5}$, CONVERGES BY COMPARISON TEST

NOTE:
A/B
- If B is made smaller, A/B gets BIGGER!
- If B is made bigger, A/B gets SMALLER!

[B] $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$

Sol: THE SERIES ______ BEHAVES LIKE ______. WHICH ______ BY ______ TEST. WE SHOW: a_n ___ b_n. NOTICE:

THEREFORE THE SERIES ______, ______ BY COMPARISON TEST

$\boxed{C}$ $\sum_{n=20}^{\infty} \frac{n^{10}+1}{n^{11}-9n^{10}-9n^{2}}$

Sol:

$\boxed{D}$ $\sum_{n=0}^{\infty} \frac{5^n}{8^n+4}$

Sol: THE SERIES $\underline{\sum \underbrace{\frac{5^n}{8^n+4}}_{a_n}}$ BEHAVES LIKE $\underline{\sum \frac{5^n}{8^n} = \sum \underbrace{(5/8)^n}_{b_n}}$ $|r| = \left|\frac{5}{8}\right| < 1$

WHICH $\underline{\text{CONVERGE}}$ BY $\underline{\text{GEO -}}$ TEST. WE SHOW:

$a_n \underline{\leq} b_n$. NOTICE :

$$a_n = \frac{5^n}{8^n+4} \leq \frac{5^n}{8^n} = b_n.$$

THEREFORE THE SERIES $\underline{\sum \frac{5^n}{8^n+4}}$, $\underline{\text{CONVERGES.}}$ BY COMPARISON TEST

$\boxed{E}$ $\sum_{n=1}^{\infty} \frac{n-6}{\sqrt{n^5+n^4+12}}$

Sol: THE SERIES $\underline{\sum \underbrace{\frac{n-6}{\sqrt{n^5+n^4+12}}}_{a_n}}$ BEHAVES LIKE $\underline{\sum \frac{n}{\sqrt{n^5}} = \sum \frac{n}{n^{5/2}} = \sum \underbrace{\frac{1}{n^{3/2}}}_{b_n}}$

WHICH $\underline{\text{CONVERGE}}$ BY $\underline{\text{P -}}$ TEST. WE SHOW:

$a_n \underline{\leq} b_n$. NOTICE :

$$a_n = \frac{n-6}{\sqrt{n^5+n^4+12}} \leq \frac{n}{\sqrt{n^5+n^4+12}} \leq \frac{n}{\sqrt{n^5}} = \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}} = b_n$$

THEREFORE THE SERIES $\underline{\sum_{n=1}^{\infty} \frac{n-6}{\sqrt{n^5+n^4+12}}}$, $\underline{\text{CONVERGES.}}$ BY COMPARISON TEST

The COMPARISON TEST uses the relationship between terms of two series. Sometimes it can be difficult (or impossible) to show that the appropriate inequality holds, in which case we CANNOT use the comparison test. But not all hope is lost, because in reality convergence or divergence of a series only depends on the long term behavior of terms as n approaches infinity.

Ex. $\sum_{n=1}^{\infty} \frac{4}{n^2-3n-5}$

$\boxed{A}$ THE SERIES $\sum \underbrace{\frac{4}{n^2-3n-5}}_{a_n}$ BEHAVES LIKE $\sum \underbrace{\frac{4}{n^2}}_{b_n}$.

WHICH CONVERGES BY P- TEST. WE SHOW: $a_n \leq b_n$. NOTICE:

$$a_n = \frac{4}{n^2-3n-5} \cancel{\leq} \frac{4}{n^2} = b_n$$

* COMPARISON IS INCONCLUSIVE

PART 2: THE LIMIT COMPARISON TEST

MAIN IDEA

If the terms of two series eventually behave the same (or at least as a constant multiple of one another) then they must both CONVERGE or both DIVERGE. To look at the eventual behavior of the terms, we take a limit of the ratio.

THM: [LIMIT COMPARISON TEST]

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C$$

Where $0 < C < \infty$, then either both series CONVERGE or both DIVERGE.

PROCEDURE

1. DETERMINE a COMPARISON SERIES: same as before.
2. DOES COMPARISON SERIES CONVERGE or DIVERGE? Use known results
3. COMPUTE the LIMIT of $\frac{a_n}{b_n}$: Use algebra to simplify. If limit is zero or infinity then this test is inconclusive. Otherwise, we can conclude that the original series behaves the same (converge/diverge) as comparison series.

Ex3. Use the **LIMIT COMPARISON TEST** to determine if the following series converge or diverge.

A $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2-3n-5}$

Sol:

THE SERIES $\sum \frac{4}{n^2-3n-5}$ BEHAVES LIKE $\sum \frac{4}{n^2}$.

WHICH CONVERGES BY P TEST. WE COMPUTE

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{4}{n^2-3n-5}}{\frac{4}{n^2}}\right) = \lim_{n\to\infty}\left(\frac{\cancel{4}}{n^2-3n-5}\cdot\frac{n^2}{\cancel{4}}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2-3n-5}\right)$$

$$= \boxed{1 = C} \quad 0<C<\infty$$

THEREFORE THE SERIES $\sum a_n$, CONVERGES BY LIMIT COMPARISON TEST

B $\displaystyle\sum_{n=1}^{\infty} \frac{n-6}{\sqrt{n^5+n^4+12}}$

Sol:

THE SERIES $\displaystyle\sum_{n=1}^{\infty} \frac{n-6}{\sqrt{n^5+n^4+12}}$ BEHAVES LIKE $\sum \frac{n}{\sqrt{n^5}} = \sum \frac{1}{n^{3/2}}$.

WHICH CONVERGES BY P- TEST. WE COMPUTE

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{n-6}{\sqrt{n^5+n^4+12}}\cdot\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty}\left(\frac{n^{5/2}-6n^{3/2}}{\sqrt{n^5+n^4+12}}\right) = 1 = c$$

$$0<C<\infty$$

THEREFORE THE SERIES $\displaystyle\sum_{n=1}^{\infty} \frac{n-6}{\sqrt{n^5+n^4+12}}$, CONVERGES BY LIMIT COMPARISON TEST

$\boxed{C}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4n}}$

Sol:

THE SERIES $\underline{\sum \frac{1}{\sqrt{n^2+4n}}}$ BEHAVES LIKE $\underline{\sum \frac{1}{\sqrt{n^2}} = \sum \frac{1}{n}}$.

WHICH DIVERGES BY P TEST. WE COMPUTE

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+4n}}\cdot\frac{n}{1}\right) = \lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+4n}}\right) = 1 = c$$

$0 < c < \infty$

THEREFORE THE SERIES $\underline{\sum a_n}$, DIVERGES BY LIMIT COMPARISON TEST

$\boxed{D}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n5^n}$

Sol: THE SERIES $\underline{\sum \frac{1}{n\cdot 5^n}}$ BEHAVES LIKE $\underline{\sum \left(\frac{1}{5}\right)^n}$.

WHICH CONVERGE BY GEO - TEST. WE COMPUTE

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{n\cdot \cancel{5^n}}\cdot\frac{\cancel{5^n}}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 = c$$

!! ☹

* Limit Comparison INCONCLUSIVE.

QUESTION: How do we know if we should use the COMPARISON TEST or the LIMIT COMPARISON TEST?

ANSWER:

- My "Go-to" is the limit comparison test because I feel as though it is more efficient and works in more instances. I usually start with this.
- If you try the limit comparison test and you get a limit of 0 or infinity, then it is inconclusive. In this case, try the comparison test.
- Ultimately, there any many examples where both will work so it comes down to a matter of preference.