TESTS for CONVERGENCE

$$\sum_{n=1}^{\infty} a_n$$

$\lim_{n\to\infty} a_n = 0$

W/ form $\sum ar^n$

GEOMETRIC SERIES TEST

- CONVERGES if $|r|<1$
- DIVERGES if $|r|\geq 1$

$a(n)=f(n)$. IF $f(n)$ ⊕ & ↓

INTEGRAL TEST

- CONVERGES OR DIVERGES WHEN $\int f(n)\,dn$ DOES.

$\lim_{n\to\infty} a_n \neq 0$

DIVERGENCE TEST

$\sum a_n$ DIVERGES

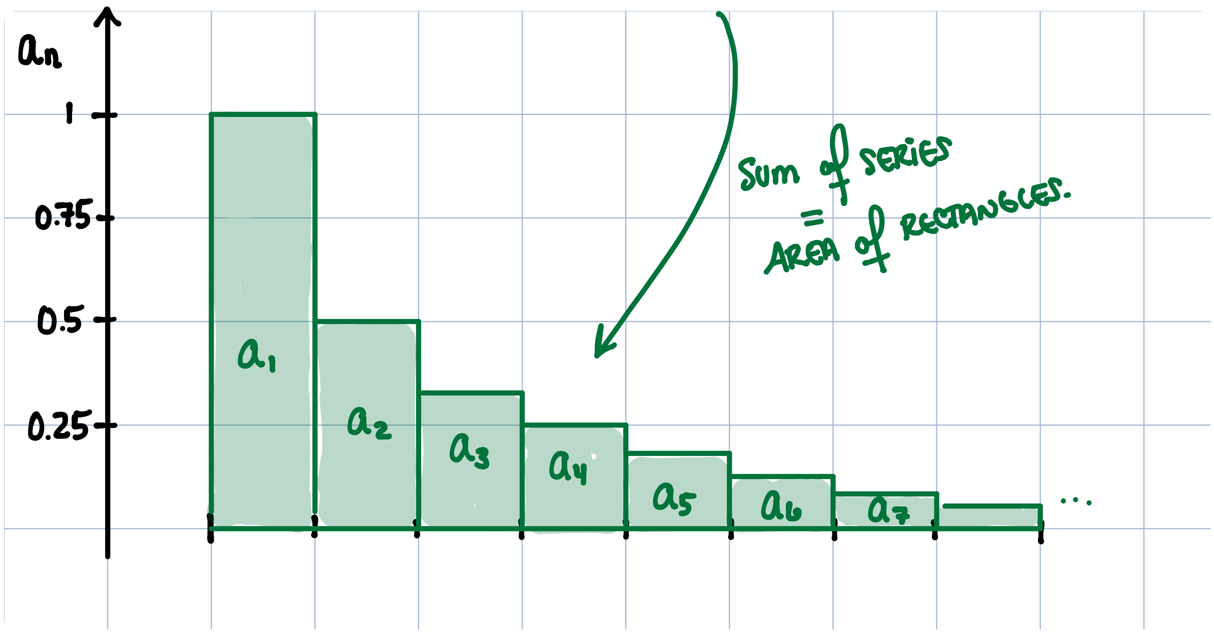
CH 11.3: THE INTEGRAL TEST

GOAL: We are going to learn another test that can be performed on certain infinite series to determine if they CONVERGE (i.e. have a fixed, finite sum) or DIVERGE (i.e. do not). This test is called the INTEGRAL TEST and it involves comparing an infinite series to an improper integral.

PART 1: VISUALIZING A SERIES

* To be able to fully understand the integral test, we first need a way to visualize a series.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$



! To find if a series converges, we need to see if this area is a fixed, finite value! We will use what we know about improper integrals.

PART 2: THE MAIN IDEA

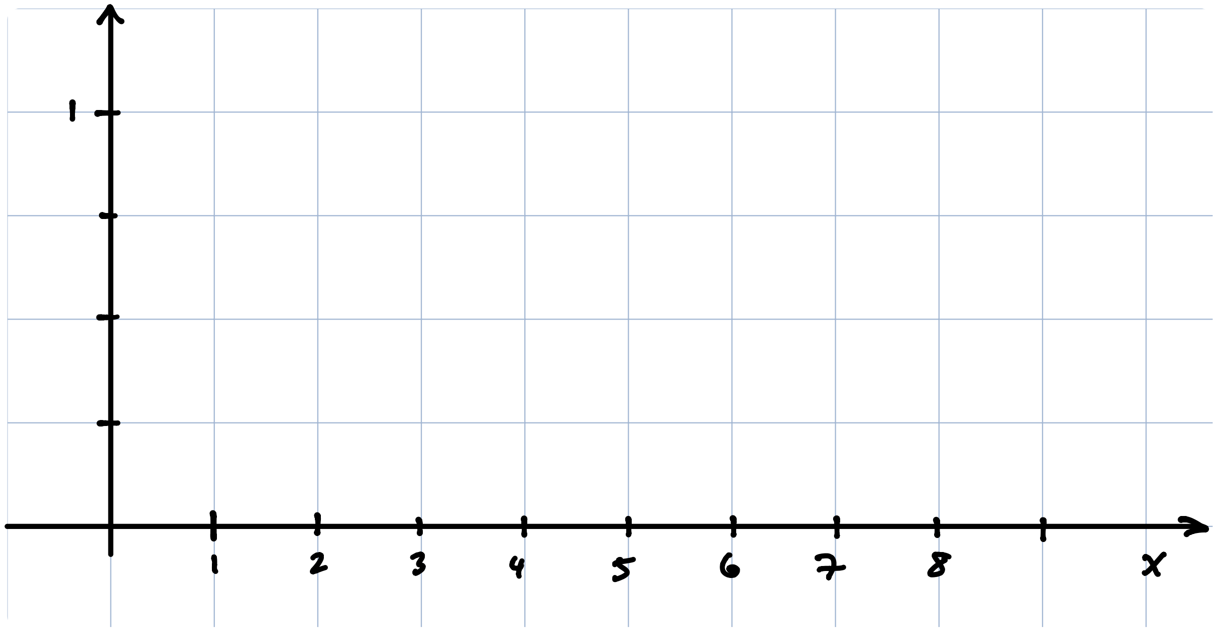
* To illustrate the idea of the INTEGRAL TEST, we will consider two separate examples.

[A] $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES

[B] $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES

A $\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{a_n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ $\quad f(x) = \frac{1}{x}$

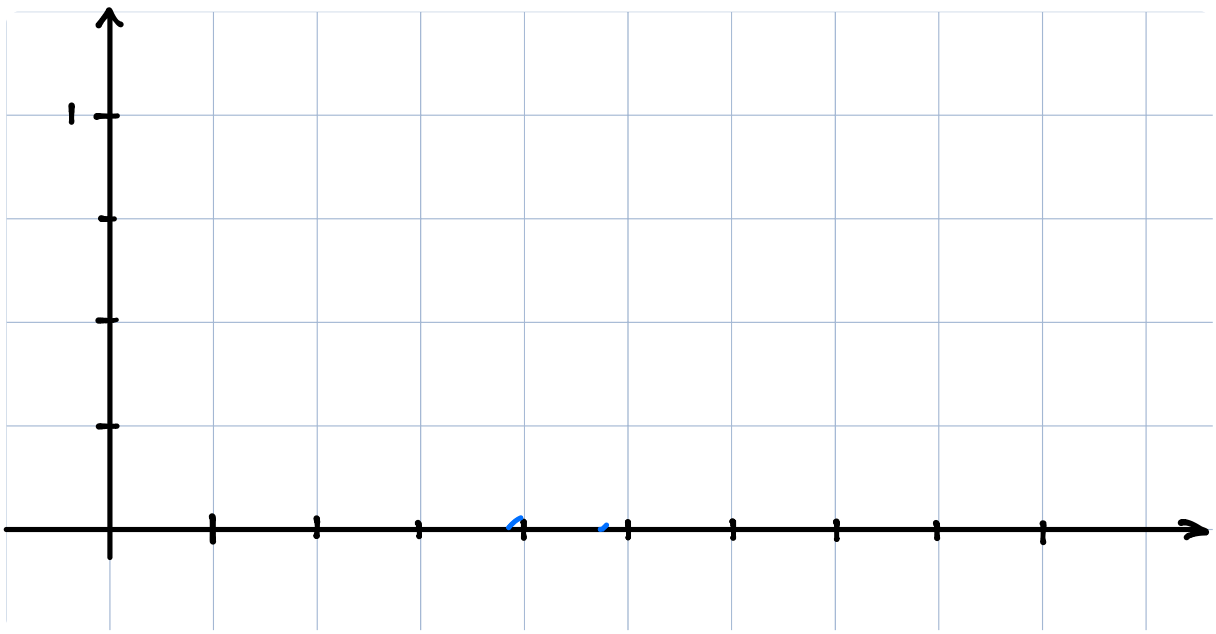
sol:



$\boxed{B}$ $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + 1/4 + 1/9 + 1/16 + 1/25 + \cdots$

$f(x) = 1/x^2$

Sol:



PART 3: THE INTEGRAL TEST

constant.

THM: [THE INTEGRAL TEST]

Suppose f is ① continuous, ② positive, & ③ decreasing on $[c, \infty)$ and let $a_n = f(n)$

(1) If $\int_c^\infty f(x)\,dx$ converges then $\sum_{n=c}^\infty a_n$ converges.

(2) If $\int_c^\infty f(x)\,dx$ diverges then $\sum_{n=c}^\infty a_n$ diverges

* You must check 3 hypothesis.

* Not necessary for $f(x)$ to always be decreasing. (Just must be eventually decreasing beyond some x value).

* Integral test does not always work!

TIPS for showing a fn is decreasing

(1) Show $f'(x) < 0$ for $x \geq c$ — does not need to be start of series.

(2) If $f(x) = \dfrac{1}{g(x)}$ and $g(x)$ is increasing for $x \geq c$.

$$\sum_{n=1}^{\infty} \frac{2n}{f(n)}$$

$f(n) = 0$

Ex 1: Test each series for CONVERGENCE or DIVERGENCE.

A $\sum_{n=1}^{\infty} \frac{2n}{n^2+2}$

Try Div. Test: $\boxed{\lim_{n\to\infty}\left(\frac{2n}{n^2+2}\right) = 0. \quad \text{Div. Test DNA.}}$

Sol: $f(x) = \frac{2x}{x^2+2} \quad x \geq 1$

① CHECK HYPOTH.

(1) f CONT ✓

(2) f POSITIVE ✓

(3) f DECREASING. ✓

$f'(x) = \frac{4-x^2}{(x^2+2)^2}$

$f'(x) < 0 \iff 4 - x^2 < 0$

$x^2 > 4$

$\underline{\underline{x > 2}}$

② COMPUTE $\int_1^{\infty} f(x)\,dx$ Ch. 7.8

$$\int_1^{\infty} \frac{2x}{x^2+2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{2x}{x^2+2}\,dx$$

SUBST $w = x^2+2$

$$\lim_{b\to\infty}\left[\ln(b^2+2) - \ln(3)\right]$$

$$= \boxed{\infty}$$

③ CONCLUSION.

$\boxed{\sum_{n=1}^{\infty} \frac{2n}{n^2+2} \text{ DIVERGES by INT. TEST}}$

B $\sum_{n=1}^{\infty} 4ne^{-2n^2}$

Sol:

STEP 1 CHECK HYPOTHESES:

☐ f POSITIVE ✓ ON $[1,\infty)$

☐ f CONTINUOUS ✓

☐ f DECREASING ✓

$f'(x) = \frac{e^{2x^2}\cdot 4 - 4x(4x)e^{2x^2}}{(e^{2x^2})^2}$

$= \frac{4-16x^2}{e^{2x^2}} \quad 4 - 16x^2 < 0$

$16x^2 > 4$

$x > 1/2$

So $f'(x) < 0$ WHEN $x > 1/2$

STEP 2: COMPUTE $\int_1^{\infty} f(x)\,dx$

$$\int_1^{\infty} 4xe^{-2x^2}\,dx \qquad w = -2x^2, \quad dw/dx = -4x$$

$$\lim_{b\to\infty}\left(-e^{-2x^2}\Big|_1^b\right) = \lim_{b\to\infty}\left(\cancel{-e^{-2b^2}}^{\to 0} + e^{-2}\right)$$

$$= e^{-2}$$

So IT CONVERGES!

STEP 3: APPLY INT. TEST.

So $\sum_{n=1}^{\infty} 4ne^{-2n^2}$ CONVERGES!

By INT TEST

C $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Sol:

Ex2. THE P-TEST FOR SERIES

RECALL: THM: [P-TEST TYPE 1] $\int_a^{\infty} \frac{1}{x^p}\,dx$ $\begin{cases} \text{CONVERGES if } p>1 \\ \text{DIVERGES if } p\le 1 \end{cases}$ FOR $a>0$

Determine values of "p" for which the series CONVERGES and for which the series DIVERGES.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad f(x) = \frac{1}{x^p},\ x\geq 1$$

Sol: ① CHECK HYPOTH.:
(1) f cont. ✓
(2) f positive ✓
(3) f DECREASING ✓
* SINCE x^p (IN DENOM) IS INCREASING.

② COMPUTE $\int_1^{\infty} \frac{1}{x^p}\,dx$ P-TEST CH 7.8

$\left[\begin{array}{l}\text{DIVERGES if } p\le 1 \\ \text{CONVERGES if } p>1\end{array}\right.$

③ CONCLUSION

THM: [P-TEST FOR SERIES]

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

CONVERGES if $p>1$

DIVERGES if $p\le 1$.

A $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $p=\frac{1}{2}$ DIVERGE

B $\sum_{n=1}^{\infty} \frac{1}{n^5}$ $p=5$ CONVERGE

PART 4: ESTIMATING THE SUM of A SERIES.

* Using the integral test, we can determine that a given series converges but we CANNOT determine what it converges to. We can, however, get a good approximation by using partial sums. Naturally, approximations are not exact. We define the REMAINDER to be the difference between the exact value of a sum and the approximate value

REMAINDER: $\boxed{R_N = S - S_N}$ (S = EXACT SUM, S_N = N^{th} PARTIAL SUM)

$$R_N = S\{(a_1+a_2+a_3+\dots) - S_N(a_1+a_2+\dots+a_N)$$

$$R_N = a_{N+1} + a_{N+2} + \dots$$

"TAIL" of SERIES.

RECALL:

$S_N = N^{th}$ PARTIAL SUM

SUM of SERIES $= S = \lim_{N\to\infty} S_N$

* Using ideas similar to the integral test, we can figure out approximately how big the remainder will be... These are called REMAINDER ESTIMATES.

THM: [REMAINDER ESTIMATES]

SUPPOSE $f(n) = a_n$ WHERE f IS CONTINUOUS, POSITIVE & DECREASING & $\sum_{n=1}^{\infty} a_n$ IS CONVERGENT.

$$\underbrace{\int_{N+1}^{\infty} f(x)\,dx}_{\text{LOWER BOUND.}} \le \underbrace{R_N}_{S-S_N} \le \underbrace{\int_{N}^{\infty} f(x)\,dx}_{\text{UPPER BOUND ON ERROR}}$$

Ex 3. Consider the series $\sum_{n=1}^{\infty} 4ne^{-2n^2}$ * CONVERGES BY EX. ABOVE.

[A] Find an explicit upper bound for the remainder R_N when estimating the series with the Nth partial sum.

Sol: $f(x) = 4xe^{-2x^2}$

* SATISFIES HYPOTHESES.

$$R_N \le \int_N^{\infty} f(x)\,dx = \text{UPPER BOUND} = e^{-2N^2}$$

$$\int_N^{\infty} 4xe^{-2x^2}\,dx = \lim_{b\to\infty}\left(\int_N^b 4xe^{-2x^2}\,dx\right) = \lim_{b\to\infty}\left(-e^{-2x^2}\Big|_N^b\right)$$

$$= \lim_{b\to\infty}\left(\cancel{-e^{-2b^2}}^{\,0} + e^{-2N^2}\right) = \boxed{e^{-2N^2}}$$

B Find an N for which the upper bound on R_N is less than 0.0001 and then compute the Nth partial sum to 5 digits for this specific N

← DESIRED ACCURACY!

Sol:

* WE WANT $R_N < 0.0001$ (ERROR)

$$R_N \leq e^{-2N^2} < 0.0001$$

SOLVE for N.

$$\ln(e^{-2N^2}) < \ln(0.0001)$$

$$-2N^2 < \ln(0.0001)$$

DIVIDE BY NEG. ⚠

$$N^2 > \frac{\ln(0.0001)}{-2} \quad \text{so} \quad N > \sqrt{\frac{\ln(0.0001)}{-2}} = 2.14$$

3rd PARTIAL SUM.

$$N = 3 \qquad \sum_{n=1}^{3} 4ne^{-2n^2} = \underbrace{4e^{-2}}_{n=1} + \underbrace{8e^{-8}}_{n=2} + \underbrace{12e^{-18}}_{n=3} = 0.544$$