CH 11.2 SERIES

PART 1: THE BASICS of SERIES

* Given a SEQUENCE we can make an INFINITE SERIES by adding all of the terms of the sequence!

SEQUENCE

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$$

SERIES

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$$

Q: BUT WAIT... how can we talk about adding an infinite number of terms together? Wouldn't this always be infinity?

A: No, we will not always get a sum of infinity. Check out these two series!

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + 25 + \ldots = \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots = 1$$

GOAL: Our job is to determine when a given series will CONVERGE (I.e. the sum is a fixed finite value) and when it will DIVERGE (I.e. the sum is not a fixed finite value).

PART 2: PARTIAL SUMS AND CONVERGENCE

* One way to determine if a series has a sum (I.e. a fixed, finite sum) is to consider PARTIAL SUMS.

SERIES: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots$

PARTIAL SUMS

$S_1 = \text{Sum of FIRST 1 TERM} = a_1$

$S_2 = \text{Sum of FIRST 2 TERMS} = a_1 + a_2$

$S_3 = \text{Sum of FIRST 3 TERMS} = a_1 + a_2 + a_3$

$S_4 = \text{Sum of FIRST 4 TERMS} = a_1 + a_2 + a_3 + a_4$

$\vdots$

SEQ. of PARTIAL SUMS:

$\{S_1, S_2, S_3, S_4, \ldots\}$

n^{th} PARTIAL SUM: $S_n = $ Sum of FIRST n TERMS $= a_1 + a_2 + a_3 + a_4 + \dots + a_n$

* In this way we create a SEQUENCE of PARTIAL SUMS $\{S_n\}$. The limit of this sequence (I.e. when n goes to infinity) will represent the SUM of the original series!

THM: [CONVERGENCE of SERIES]:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$, let S_n denote the nth partial sum:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

If the sequence $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = s$ exists as a real number, then the series $\sum a_n$ is called CONVERGENT and we write:

$$\sum_{n=1}^{\infty} a_n = s \quad \text{OR} \quad a_1 + a_2 + a_3 + \dots = s$$

The number s is called the sum of the series. If the sequence $\{S_n\}$ is divergent, then the series is DIVERGENT.

Ex 2 Do the PARTIAL SUMS for the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ ($a_1 + a_2 + a_3 + a_4$)
Support the fact the sum is 1 (as stated in the previous section)?

sol:

$a_n = \frac{1}{2^n}$

$\lim_{n\to\infty} a_n = 0$

PARTIAL SUMS of SERIES.

n	Sum of first n terms	
1	0.50000000	$= a_1$ (S_1)
2	0.75000000	$= a_1 + a_2$
3	0.87500000	$= a_1 + a_2 + a_3$
4	0.93750000	
5	0.96875000	
6	0.98437500	
7	0.99218750	
10	0.99902344	
15	0.99996948	
20	0.99999905	
25	0.99999997	

$$\lim_{n\to\infty} S_n = 1 = \sum_{n=1}^{\infty} a_n$$

Ex 3. Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given by:

A $S_n = \frac{4n+3}{8n+1}$

sol:

Sum of SERIES $= \sum_{n=1}^{\infty} a_n =$ limit of PARTIAL SUMS

$= \lim_{n\to\infty} S_n = \boxed{\frac{1}{2}}$

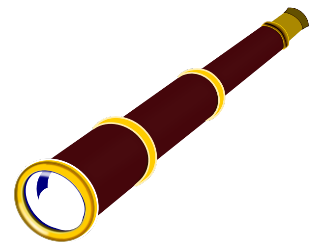
B $S_n = 5 - (0.2)^n$

sol:

Sum of SERIES $= \sum_{n=1}^{\infty} a_n =$ limit of PARTIAL SUMS

$= \lim_{n\to\infty} S_n = \boxed{5}$

Part 3: TELESCOPING SUMS



* Some series have a special property that the terms of the series cancel each other out. When this happens we have what is called a TELESCOPING SERIES and it is easy for us to find the sum. Check it out:

Ex 4: Find the sum of each series by determining a formula for the Nth partial sum and taking a limit:

S_N

A $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Sol:

$$S_N = \text{sum of first } N \text{ terms} = \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+1}$$

$$= \left(1+\cancel{\tfrac{1}{2}}+\cancel{\tfrac{1}{3}}+\cancel{\tfrac{1}{4}}+\dots+\cancel{\tfrac{1}{N}}\right)-\left(\cancel{\tfrac{1}{2}}+\cancel{\tfrac{1}{3}}+\cancel{\tfrac{1}{4}}+\dots+\cancel{\tfrac{1}{N}}+\tfrac{1}{N+1}\right)$$

So $\boxed{S_N = 1-\frac{1}{N+1}}$ N^{th} partial sum.

$$\text{sum of series} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \underset{N\to\infty}{\text{limit}} \text{ of } S_N = \lim_{N\to\infty}\left(1-\frac{1}{N+1}\right) = \boxed{1}$$

B $\sum_{n=1}^{\infty}\frac{2}{n(n+2)}$ * use Partial Fracs First.

Sol:

PART 4: GEOMETRIC SERIES

Ex. $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$7 + \frac{35}{3} + \frac{175}{9} + \cdots$

* An important type of infinite series is called a GEOMETRIC SERIES in which the ratio between subsequent terms is constant.

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \cdots}$$

RATIO

1st TERM

! CAN ALSO BE WRITTEN $\sum_{n=0}^{\infty} ar^n$

THM: [CONVERGENCE of GEO. SERIES]

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{CONVERGES TO } \boxed{\frac{a}{1-r}} & \text{if } |r|<1 \\ \text{DIVERGE} & \text{if } |r| \geq 1 \end{cases}$$

$\left(\sum_{n=0}^{\infty} ar^n\right)$

Pf:

pf

Ex 2. Determine if the following GEOMETRIC SERIES converge or diverge. If they converge, find the SUM of the series.

$\sum_{n=0}^{\infty} ar^n$

A $\sum_{n=1}^{\infty} \frac{3}{2^n} = \sum_{n=1}^{\infty} 3\cdot\left(\frac{1}{2}\right)^n = \frac{3}{2} + \frac{3}{4} + \cdots$

Sol: $a = \boxed{3/2}$

$r = \frac{\text{2nd TERM}}{\text{1st TERM}} = \frac{3/4}{3/2} = \boxed{\frac{1}{2}}$

So $|r| < 1$ so $\sum \frac{3}{2^n}$ CONVERGES TO $\frac{3/2}{1-\frac{1}{2}}$

B $\sum_{n=1}^{\infty} 3\cdot 2^n = 6 + 12 + \cdots$

Sol: $a = \boxed{6}$

$r = \frac{12}{6} = \boxed{2}$

$|r| \geq 1$ so $\sum_{n=1}^{\infty} 3\cdot 2^n$ DIVERGES

C $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{2^{n+3}} = \frac{5}{16} + \frac{25}{32} + \cdots$

Sol:

$a = \boxed{\frac{5}{16}}$

$r = \dfrac{25/32}{5/16} = \dfrac{25}{32} \cdot \dfrac{16}{5} = \boxed{\frac{5}{2}}$

$|r| \geq 1$ so DIVERGES

D $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{2^{3n+1}} = \sum \frac{6^n}{2 \cdot 2^{3n}} = \sum \frac{6^n}{2 \cdot (2^3)^n}$ * ALT. METHOD !

Sol:

$= \displaystyle\sum_{n=0}^{\infty} \frac{6^n}{2 \cdot 8^n} = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(\frac{6}{8}\right)^n$

$= \displaystyle\sum_{n=0}^{\infty} \underbrace{\frac{1}{2}}_{a} \underbrace{\left(\frac{3}{4}\right)}_{r}^n$

$|r| < 1$ so $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{2^{3n+1}}$ CONVERGES TO $\dfrac{1/2}{1 - 3/4}$

Ex 6: Find the values of x for which the series converges. For those values, find the sum in terms of x.

A $\displaystyle\sum_{n=0}^{\infty} 1(3x)^n$ *GEOMETRIC : $\displaystyle\sum_{n=0}^{\infty} ar^n$

Sol: $a = 1$ $r = 3x$

CONVERGE if $|r| < 1$

$|3x| < 1$

So $3|x| < 1$

$|x| < \frac{1}{3} \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$ VALUES of X WHEN SERIES CONVERGES.

Sum $= \dfrac{a}{1-r} = \boxed{\dfrac{1}{1-3x}}$

Ex 7. Using the aid of a GEOMETRIC SERIES, express the decimal as a ratio of integers (in reduced form)

Sol:

$0.8\overline{27} = 0.827272727\ldots$

$= 0.8 + \dfrac{27}{1000} + \dfrac{27}{100000} + \dfrac{27}{10000000} + \cdots$

$= 0.8 + \underbrace{\dfrac{27}{10^3} + \dfrac{27}{10^5} + \dfrac{27}{10^7} + \cdots}_{\text{GEO SERIES.}}$

sum $a = 27/10^3$ $r = \dfrac{1}{10^2}$

$= 0.8 + \dfrac{a}{1-r} = 0.8 + \dfrac{27/1000}{1 - 1/100} = \dfrac{8}{10} + \dfrac{27}{990} = \boxed{\dfrac{819}{990}}$

PART 5: DIVERGENCE TEST and PROPERTIES of SERIES

* Series can be broken down into two main categories based on the behavior of the terms of the series: $\sum_{n=1}^{\infty} a_n$

$\lim_{n\to\infty} a_n = 0$ (I.e. terms APPROACH ZERO)	$\lim_{n\to\infty} a_n \neq 0$ (I.e. terms DO NOT APPROACH ZERO)
When the terms of a series approach zero, we CANNOT immediately conclude that the series converges. There are some cases where it still might diverge (see example below) so we need to apply TESTS FOR CONVERGENCE to determine if it converges. We already learned one test for geometric series and we will learn more in this chapter.	When the terms of a series do not approach zero, this means that as n goes to infinity, our sum will continue to change and thus the series cannot converge! This leads to an important theorem:

Ex. [HARMONIC SERIES]

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

DIVERGES

THM: [THE DIVERGENCE TEST]:

if $\lim_{n\to\infty} a_n \neq 0$ THEN $\sum_{n=1}^{\infty} a_n$ will DIVERGE

! if A SERIES DIVERGES THEN WE DON'T NECESSARILY KNOW THAT

$$\lim_{n\to\infty} a_n \neq 0.$$

Ex 8. If possible, determine if the following series are CONVERGENT OR DIVERGENT and explain why.

A $\displaystyle\sum_{n=1}^{\infty} \frac{4n+1}{3n+5}$ a_n

Sol: $\displaystyle\lim_{n\to\infty} a_n = \boxed{\tfrac{4}{3}} \neq 0$

$\displaystyle\sum_{n=1}^{\infty} a_n$ DIVERGES By DIV. TEST

B $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ a_n

Sol: $\displaystyle\lim_{n\to\infty} a_n = \boxed{0}$

* DIVERGENCE TEST DOES NOT APPLY. WE KNOW NOTHING (YET!)

C $\displaystyle\sum_{n=1}^{\infty} 4e^n$ a_n

Sol: $\displaystyle\lim_{n\to\infty} 4e^n = \boxed{\infty} \neq 0$

So $\displaystyle\sum_{n=1}^{\infty} a_n$ DIVERGES By DIV. TEST.

** Finally, there are some important PROPERTIES of series that we should be familiar with:

THM: If $\sum a_n$ and $\sum b_n$ are convergent series then so are the series:

$$\sum_{n=1}^{\infty} c a_n, \quad \sum_{n=1}^{\infty} (a_n + b_n), \quad \text{and} \quad \sum_{n=1}^{\infty} (a_n - b_n)$$

Furthermore,

- $\displaystyle\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$
- $\displaystyle\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

PROPERTY: A finite number of terms does not affect the convergence of a series. Effectively, this means that it does not matter where the series starts.

$$\sum_{n=1}^{\infty} \frac{3}{2^n} \qquad \sum_{n=1}^{\infty} a\, r^{n-1} \quad \text{OR} \quad \sum_{n=0}^{\infty} a\, r^{n}$$

$$= \sum_{n=1}^{\infty} 3 \cdot \left(\tfrac{1}{2}\right)^n = \sum_{n=1}^{\infty} 3\left(\tfrac{1}{2}\right) \cdot \left(\tfrac{1}{2}\right)^{n-1}$$

$a = \frac{3}{2}$ $\qquad r = \frac{1}{2}$