CH 11.2 SERIES

PART 1: THE BASICS of SERIES

* Given a SEQUENCE we can make an INFINITE SERIES by adding all of the terms of the sequence!

SEQUENCE

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$$

$\Rightarrow$

SERIES

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$$

Q: BUT WAIT... how can we talk about adding an infinite number of terms together? Wouldn't this always be infinity?

A: No, we will not always get a sum of infinity. Check out these two series!

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + 25 + \ldots$$
$$= \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$$
$$= 1$$

GOAL: Our job is to determine when a given series will CONVERGE (I.e. the sum is a fixed finite value) and when it will DIVERGE (I.e. the sum is not a fixed finite value).

PART 2: PARTIAL SUMS AND CONVERGENCE

* One way to determine if a series has a sum (I.e. a fixed, finite sum) is to consider PARTIAL SUMS.

SERIES: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots$

PARTIAL SUMS

$$\left\{\begin{array}{l} S_1 = \text{Sum of FIRST 1 TERM} = a_1 \\ S_2 = \text{Sum of FIRST 2 TERMS} = a_1 + a_2 \\ S_3 = \text{Sum of FIRST 3 TERMS} = a_1 + a_2 + a_3 \\ S_4 = \text{Sum of FIRST 4 TERMS} = a_1 + a_2 + a_3 + a_4 \\ \vdots \end{array}\right.$$

n^{th} PARTIAL SUM $\rightarrow$ $S_n = $ Sum of FIRST n TERMS $= a_1 + a_2 + a_3 + a_4 + \dots + a_n$

* In this way we create a SEQUENCE of PARTIAL SUMS $\{S_n\}$. The limit of this sequence (I.e. when n goes to infinity) will represent the SUM of the original series!

THM: [CONVERGENCE of SERIES]:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$, let S_n denote the nth partial sum:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

If the sequence $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is called CONVERGENT and we write:

$$\sum_{n=1}^{\infty} a_n = S \quad \text{OR} \quad a_1 + a_2 + a_3 + \dots = S$$

The number s is called the sum of the series. If the sequence $\{S_n\}$ is divergent, then the series is DIVERGENT.

Ex 2 Do the PARTIAL SUMS for the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ Support the fact the sum is 1 (as stated in the previous section)?

sol:

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

Ex 3. Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given by:

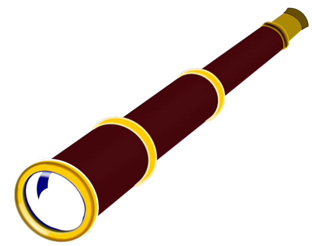
[A] $S_n = \frac{4n+3}{8n+1}$

sol:

[B] $S_n = 5 - (0.2)^n$

sol:

Part 3: TELESCOPING SUMS



* Some series have a special property that the terms of the series cancel each other out. When this happens we have what is called a TELESCOPING SERIES and it is easy for us to find the sum. Check it out:

Ex 4: Find the sum of each series by determining a formula for the Nth partial sum and taking a limit:

A $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Sol:

B $\sum_{n=3}^{\infty}\frac{8}{n^2-4}$

Sol:

PART 4: GEOMETRIC SERIES

* An important type of infinite series is called a GEOMETRIC SERIES in which the ratio between subsequent terms is constant.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \dots$$

THM: [CONVERGENCE of GEO. SERIES]

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \\ \\ \end{cases}$$

$(\qquad)$

Pf:

Ex 2. Determine if the following GEOMETRIC SERIES converge or diverge. If they converge, find the SUM of the series.

[A] $\sum_{n=1}^{\infty} \frac{3}{2^n}$

Sol:

[B] $\sum_{n=1}^{\infty} 3\cdot 2^n$

Sol:

$\boxed{C}$ $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{2^{n+3}}$

Sol:

$\boxed{D}$ $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{2^{3n+1}}$

Sol:

Ex 6: Find the values of x for which the series converges. For those values, find the sum in terms of x.

$\boxed{A}$ $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

Sol:

Ex 7. Using the aid of a GEOMETRIC SERIES, express the decimal as a ratio of integers (in reduced form)

$$0.8\overline{27} = 0.827272727\ldots$$

Sol:

PART 5: DIVERGENCE TEST and PROPERTIES of SERIES

* Series can be broken down into two main categories based on the behavior of the terms of the series:

$$\sum_{n=1}^{\infty} a_n$$

$\lim_{n\to\infty} a_n = 0$ (I.e. terms APPROACH ZERO)	$\lim_{n\to\infty} a_n \neq 0$ (I.e. terms DO NOT APPROACH ZERO)
When the terms of a series approach zero, we CANNOT immediately conclude that the series converges. There are some cases where it still might diverge (see example below) so we need to apply TESTS FOR CONVERGENCE to determine if it converges. We already learned one test for geometric series and we will learn more in this chapter.	When the terms of a series do not approach zero, this means that as n goes to infinity, our sum will continue to change and thus the series cannot converge! This leads to an important theorem:

Ex. [HARMONIC SERIES]

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

THM: [THE DIVERGENCE TEST]:

!

Ex 8. If possible, determine if the following series are CONVERGENT OR DIVERGENT and explain why.

A $\sum_{n=1}^{\infty} \frac{4n+1}{3n+5}$

Sol:

B $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Sol:

C $\sum_{n=1}^{\infty} 4e^{n}$

Sol:

** Finally, there are some important PROPERTIES of series that we should be familiar with:

THM: If $\sum a_n$ and $\sum b_n$ are convergent series then so are the series:

$$\sum_{n=1}^{\infty} ca_n, \quad \sum_{n=1}^{\infty} (a_n+b_n), \quad \text{and} \quad \sum_{n=1}^{\infty} (a_n-b_n)$$

Furthermore,

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

PROPERTY: A finite number of terms does not affect the convergence of a series. Effectively, this means that it does not matter where the series starts.