CH 11.1: SEQUENCES

PART 1: THE BASICS

DEFN: [SEQUENCE]: * n IS INTEGER.

A LIST of #'s w/ A DEFINED ORDER

$$\{a_1, a_2, a_3, \dots, a_n, \dots\}$$

↑ First TERM ↑ GENERAL TERM.

NOTATION

There are 3 ways to express a SEQUENCE

① LIST

$\{a_1, a_2, a_3, \dots\}$ — Sometimes WE START w/ a_0

② FORMULA:

$a_n = \square \ , n \geq 1$

③ SEQ NOTATION $\{a_n\}_{n=1}^{\infty}$

Ex 1. [FINDING TERMS of A SEQUENCE]

For each of the following sequences, give the formula for the nth term of the sequence and write out the first several terms of the sequence.

A $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$, $a_n = \frac{n+1}{n}$, $n \geq 1$, $\{2, 3/2, 4/3, 5/4, \dots\}$

$n=1 \quad n=2 \quad n=3 \quad n=4.$

B $\{(-1)^n \cdot n^2\}_{n=0}^{\infty}$, $a_n = (-1)^n \cdot n^2 \quad n \geq 0$ $\{0, -1, 4, -9, \dots\}$

$n=0 \quad n=1 \quad n=2 \quad n=3$

C $\{\sqrt{n-2}\}_{n=2}^{\infty}$, $a_n = \sqrt{n-2} \quad n \geq 2$

$\{0, 1, \sqrt{2}, \sqrt{3}, 2, \dots\}$

! ALTERNATING SEQUENCES.

$(-1)^n$: ODD TERMS NEGATIVE

$(-1)^{n+1}$: EVEN TERMS NEGATIVE.

D $\{\sin(n\pi)\}_{n=1}^{\infty}$ $a_n = \sin(n\pi) \quad n \geq 1$

$\{0, 0, 0, 0, \dots\}$

Ex 2. [FINDING a_n]

For each of the following sequences, give a formula for the general term a_n of the sequence, assuming that the given pattern continues.

A $\{2, 4, 8, 16, 32, 64 \dots\}$

Sol: START $n=1$ $\boxed{\{2^n\}_{n=1}^{\infty}}$

START $n=0$ $\boxed{\{2^{n+1}\}_{n=0}^{\infty}}$

! HOW ABOUT $\{1, 2, 4, 8, 16 \dots\}$

START $n=1$ $\{2^{n-1}\}_{n=1}^{\infty}$

START $n=0$ $\{2^n\}_{n=0}^{\infty}$

$\boxed{B}$ $\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots\right\}$

Sol: $\underset{n=1}{1}$

$$\boxed{\left\{\frac{(-1)^{n+1}}{n}\right\}_{n=1}^{\infty}}$$

$\boxed{C}$ $\left\{\underset{n=1}{\frac{4}{3}}, \underset{n=2}{\frac{5}{9}}, \underset{n=3}{\frac{6}{27}}, \underset{n=4}{\frac{7}{81}}, \ldots\right\}$

Sol: $n=1$

$$\boxed{\left\{\frac{n+3}{3^n}\right\}_{n=1}^{\infty}}$$

$\boxed{D}$ $\left\{-\frac{7}{2}, \frac{7}{5}, -\frac{7}{8}, \frac{7}{11}, -\frac{1}{2}, \ldots\right\}$ $-\frac{7}{14}$

Sol:

$$\boxed{\left\{\frac{(-1)^n\, 7}{3n-1}\right\}_{n=1}^{\infty}}$$

* Common Patterns:

2^n	$2, 4, 8, 16, 32, \ldots$
3^n	$3, 9, 27, 81, \ldots$
5^n	$5, 25, 125, 625, \ldots$

NOTE: The formulae found above are referred to as the **CLOSED FORM** of the respective sequences. Not all sequences have a closed form.

Ex. $\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$ $\quad e^1 = 2.\underbrace{718281828}45\ldots$

** Some sequences can be defined by relating each term to the preceding terms of the sequence. This is called a **RECURSIVE SEQUENCE.**

DEFN: [RECURSIVE SEQUENCE]

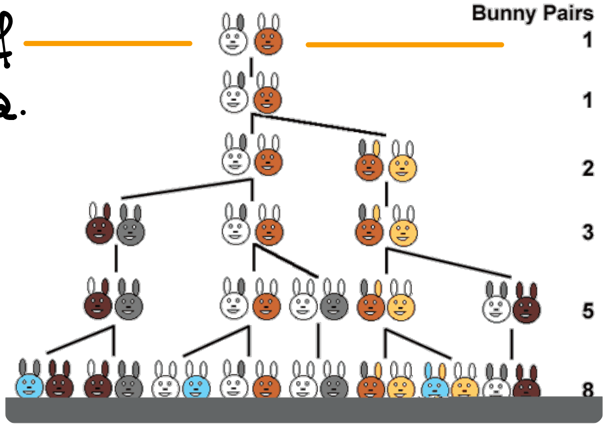
A seq for which a_n can be expressed as a function of the preceding terms. $(a_{n-1}, a_{n-2}, \text{etc})$.

Ex. Fibonacci Seq

$$a_n = a_{n-1} + a_{n-2} \quad n \geq 3$$

w/ $a_1 = 1$ & $a_2 = 1$.

THE ORIGIN of FIBONACCI SEQ.



Ex 3. [FINDING TERMS of A RECURSIVE SEQ]. Find the first 4 terms of each sequence:

$\boxed{A}$ $a_n = 3a_{n-1} + 2$ for $n \geq 2$
$a_1 = 1$

Sol:

$a_1 = 1$
$a_2 = 3a_1 + 2 = 5$
$a_3 = 3a_2 + 2 = 17$
$a_4 = 3a_3 + 2 = 53$
$\vdots$

$\boxed{B}$ $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$
$a_1 = 1 \quad a_2 = 2$

Sol:

$a_1 = 1$
$a_2 = 2$
$a_3 = a_2 + 2a_1 = 4$
$a_4 = a_3 + 2a_2 = 8$
$\vdots$

! SOME RECURSIVE SEQS HAVE CLOSED forms.
$\{2^n\}_{n=0}^{\infty}$

Ex 4. [FINDING RECURSIVE FORMULA] Find a recursive formula for each sequence:

$\boxed{A}$ $\{1, 2, 6, 24, 120, \ldots\}$
×2 ×3 ×4 ×5

Sol: n=1, n=2, n=3, n=4, n=5

$$\boxed{\begin{array}{l} a_n = n \cdot a_{n-1} \quad n \geq 2 \\ a_1 = 1 \end{array}}$$

$\boxed{B}$ $\{1, 3, 7, 15, 31, \ldots\}$

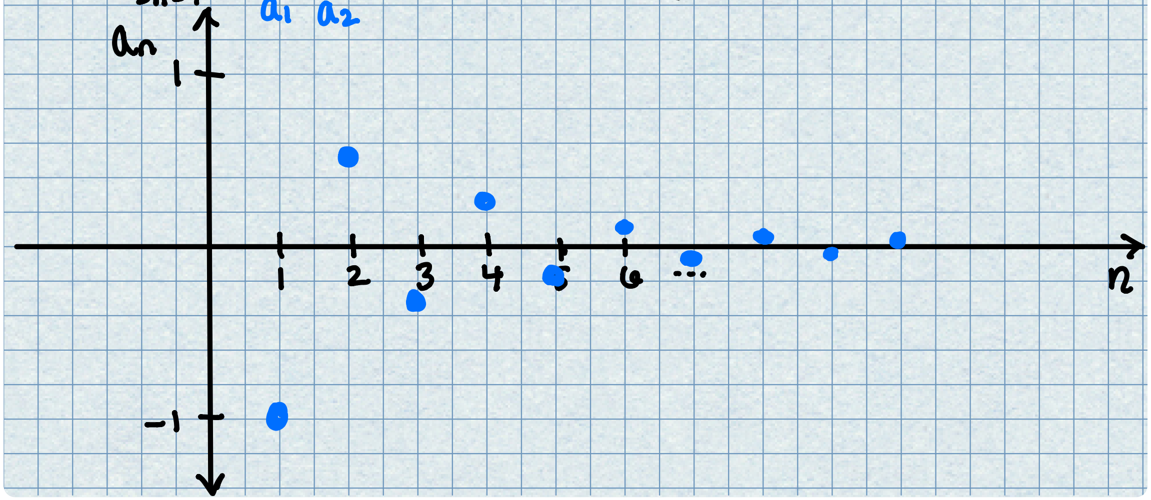
Sol:

$$\boxed{\begin{array}{l} a_n = 2a_{n-1} + 1 \quad n \geq 2 \\ a_1 = 1 \end{array}}$$

PART 2: HOW TO VISUALIZE A SEQUENCE

* Plot n on x-axis and a_n on y-axis.

$$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty} = \left\{\underset{a_1}{-1}, \underset{a_2}{\tfrac{1}{2}}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \ldots\right\}$$



We could also just plot the terms on a number line, but this is usually not very helpful.

PART 3: THE LIMIT of A SEQUENCE.

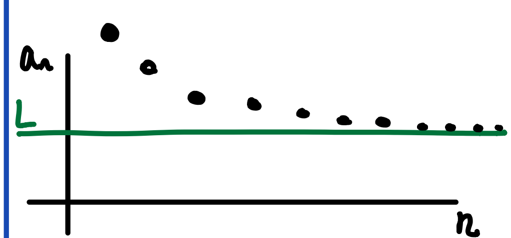
* We often want to know what happens to the terms of a sequence as n goes to infinity. This is called the **LIMIT OF THE SEQUENCE** and is denoted by:

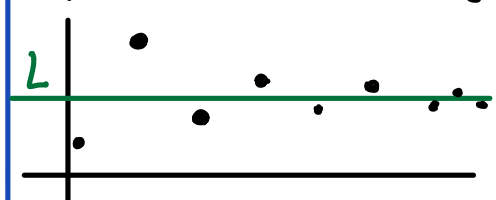
DEFN: [LIMIT of A SEQUENCE]

A SEQ $\{a_n\}$ HAS A LIMIT "L"

$$\lim_{n\to\infty}(a_n) = L \qquad (a_n \longrightarrow L)$$

IF THE TERMS a_n CAN BE MADE AS CLOSE TO L AS WE LIKE BY TAKING n LARGE ENOUGH. IF IT EXISTS, THE SEQ CONVERGES (i.e IS CONVERGENT) OTHERWISE it DIVERGES





Q: WHEN IS A SEQ DIVERGENT?

A: If IT DOES NOT APPROACH 1 FINITE VALUE.

NOTE: A more precise definition of this is given next:

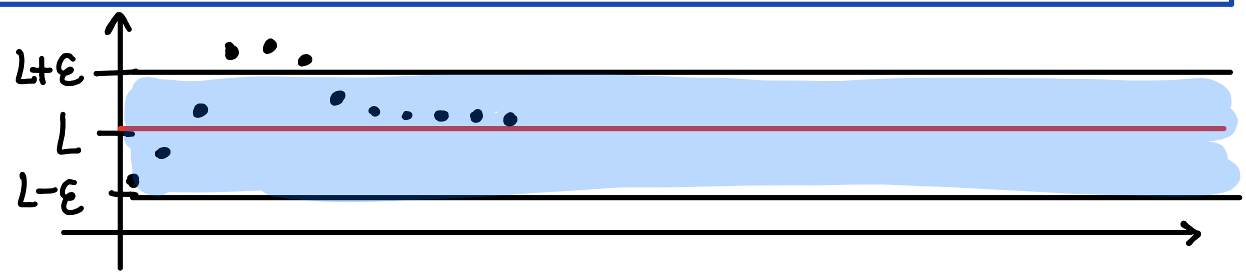
DIVERGES $\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, 1, \ldots\}$

DEFN: [LIMIT of A SEQUENCE]

"ε - DEFN"

A SEQ $\{a_n\}$ CONVERGES TO A Limit "L" if for every $\varepsilon > 0$ THERE IS AN INTEGER N SUCH THAT

if $n > N$ $\quad |a_n - L| < \varepsilon$



To compute limits of sequences, all of the rules that we had for functions still apply. That is:

"n" IS THE NEW "x"

PROPERTIES of LIMITS

$\{a_n\}, \{b_n\}$ CONV. SEQ's.

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$$

* provided $b_n \not\to 0$

POWER

$$\lim_{n\to\infty}(a_n)^p = \left(\lim_{n\to\infty}(a_n)\right)^p$$

RULES for LIMITS

① $$\lim_{n\to\infty}(r^n) = \begin{cases} 1 & \text{if } r=1 \\ 0 & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r|>1 \\ \text{DNE.} & \text{if } r=-1 \end{cases}$$

r = constant

② $f(n), g(n)$ POLYNOMIALS.

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \begin{cases} \text{DIVERGE} & \text{if } \deg f > \deg g \\ 0 & \text{if } \deg f < \deg g \\ \text{RATIO of Coeff} & \text{if } \deg f = \deg g \end{cases}$$

③ L'HOPITAL'S RULE ⤵.

Ex 5. [CONVERGENCE of SEQUENCES].

Determine if the following sequences CONVERGE or DIVERGE and provide justification. If it converges, find the limit. It may be helpful to plot a few points of the sequence.

A $\left\{\left(\frac{2}{3}\right)^n\right\}_{n=0}^{\infty}$

sol: $a_n = \left(\frac{2}{3}\right)^n$ ← r

$$\boxed{\lim_{n\to\infty} a_n = 0}$$ SINCE $|r| < 1$.

B $\left\{\left(\frac{3}{2}\right)^n\right\}_{n=0}^{\infty}$

sol: $a_n = \left(\frac{3}{2}\right)^n$ ← r

$$\boxed{\lim_{n\to\infty} a_n = \infty}$$ $\{a_n\}$ DIVERGES

SINCE $|r| > 1$.

C $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

sol: $a_n = 1/n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) = \boxed{0}$$

a_n CONVERGES TO

D $\left\{\frac{5+2n^2}{4+8n^2}\right\}_{n=1}^{\infty}$

sol: $\lim_{n\to\infty}\left(\frac{5+2n^2}{4+8n^2}\right) = \boxed{\frac{1}{4}}$ $\{a_n\}$ CONVERGES TO

E $\left\{\frac{n^2+1}{\sqrt{n^3+4}}\right\}_{n=0}^{\infty}$

sol: $\lim_{n\to\infty}\left(\frac{\overset{\text{DEG } 2}{n^2}+1}{\underset{\text{DEG } 3/2}{\sqrt{n^3}}+4}\right) = \boxed{\infty}$ $\{a_n\}$ DIVERGES

F $\left\{\frac{e^n+1}{e^n-1}\right\}_{n=1}^{\infty}$

sol: $\lim_{n\to\infty}\left(\frac{e^n+1}{e^n-1}\right) \overset{\frac{\infty}{\infty}}{\underset{\text{L.R.}}{\longrightarrow}} \lim_{n\to\infty}\left(\frac{e^n}{e^n}\right) = \boxed{1}$ $\{a_n\}$ CONV. TO

G $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

sol:

THM: [LIMITS IN CONTINUOUS fn's]

If $\lim\limits_{n\to\infty}(a_n) = L$ AND if f IS CONTINUOUS

THEN

$$\lim_{n\to\infty} f(a_n) = f\Big(\underbrace{\lim_{n\to\infty} a_n}\Big) = f(L).$$

G $\{e^{4/n}\}_{n=1}^{\infty}$

sol:

$$\lim_{n\to\infty}\left(e^{4/n}\right) = e^{\lim_{n\to\infty}(4/n)}$$
$$= \boxed{e^0}$$

H $\left\{\ln\left(\frac{n^2+1}{n^2+6}\right)\right\}_{n=1}^{\infty}$

sol:

Limits of ALTERNATING SEQUENCES can be tricky, as evidenced by the next few examples.

I $\{\cos(n\pi)\}_{n=1}^{\infty}$

sol: $\{-1, 1, -1, 1, -1, 1, \ldots\}$

DIVERGES

THM: [CONVERGENCE of ALTERNATING SEQUENCES]

If $\lim\limits_{n\to\infty} |a_n| = 0$ THEN

ALTERNATING SEQUENCE

$$\lim_{n\to\infty}(a_n) = 0$$

OTHERWISE $\{a_n\}$ DIVERGES!

J $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

sol:

$\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\}$

CONVERGES TO 0

$\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = 0$ so $\left\{\frac{(-1)^n}{n}\right\}$ conv. to 0.

K $\left\{\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right\}_{n=1}^{\infty}$

sol:

PART 4: SOME CHARACTERISTICS of SEQUENCES

DEFN: [MONOTONIC SEQUENCES]

for A SEQ $\{a_n\}$, $\{a_n\}$ is:

- **INCREASING**: if $a_n > a_{n-1}$ for ALL n. $(a_1 < a_2 < a_3 < a_4 < \dots)$
- **DECREASING** if $a_n < a_{n-1}$ for ALL n $(a_1 > a_2 > a_3 > a_4 \dots)$

* if $\{a_n\}$ IS INCR. OR DECR WE SAY it's MONOTONIC!

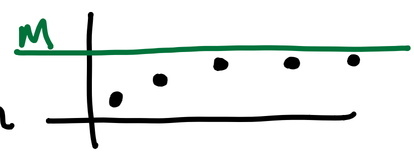
There are two ways to determine if a sequence is INCREASING or DECREASING (or neither).

- Check conditions from definition. OR
- Let $f(n) = a_n$ and compute $f'(n)$. Beginning of SEQ.
  - If $f'(n) > 0$ for $n > c$ then the sequence is INCREASING
  - If $f'(n) < 0$ for $n > c$ then the sequence is DECREASING

DEFN: [BOUNDED SEQUENCES]

$\{a_n\}$ IS

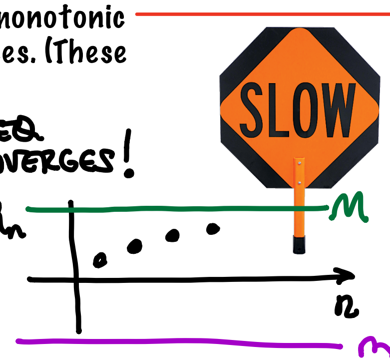
- **BOUNDED ABOVE** if THERE EXISTS A # M SUCH THAT $a_n \leq M$ for ALL n



- **BOUNDED BELOW** if THERE EXISTS A # m s.t $a_n \geq m$ for all n

* if $\{a_n\}$ IS BOUNDED ABOVE + BELOW WE SAY it is BOUNDED.

NOTE: There are some very important connections between monotonic sequences, bounded sequences, and convergent sequences. (These are great TRUE/FALSE questions)



MONOTONIC SEQ THM: Every BOUNDED, MONOTONIC SEQ CONVERGES!

(1) NOT ALL CONVERGENT SEQ's ARE MONOTONE. $\left(\left\{\frac{(-1)^n}{n}\right\}\right.$ CONVERGES TO 0$\left.\right)$

(2) NOT ALL MONOTONE SEQ's ARE CONVERGENT. $(\{e^n\})$

(3) NOT ALL BOUNDED SEQ's CONVERGE. $(\{(-1)^n\})$

(4) ALL INCREASING SEQ's ARE BOUNDED BELOW BY THE FIRST TERM.

(5) ALL DECREASING SEQ's ARE BOUNDED ABOVE BY FIRST TERM.

Ex 6. [INCR/DECR/BOUNDED]

Find the first 4 terms of each sequence, plot a_n vs. n, and then determine if each is INCREASING, DECREASING, or neither. Also determine if the sequence is BOUNDED.

[A] $a_n = \frac{2n-1}{3n+2}$, $n \geq 1$ $\left\{\frac{1}{5}, \frac{3}{8}, \frac{5}{11}, \frac{7}{14}, \frac{9}{17} \cdots\right\}$

Sol: STEP 1: INCR/DECR:

$$f(n) = \frac{2n-1}{3n+2} \quad n \geq 1$$

$$f'(n) = \frac{7}{(3n+2)^2} > 0 \quad \text{So}$$

$\{a_n\}$ IS INCREASING.! $(\uparrow)$

STEP 2: CHECKING BOUNDEDNESS

* SINCE $\{a_n\}$ is $\uparrow$ THEN IT'S BOUNDED BELOW BY $\boxed{m = \frac{1}{5}}$

* TO SEE IF IT'S BOUNDED ABOVE. TAKE LIMIT.

$$\lim_{n\to\infty}\left(\frac{2n-1}{3n+2}\right) = \boxed{\frac{2}{3} = M}$$

So $\{a_n\}$ IS BOUNDED

$$\frac{1}{5} \leq a_n \leq \frac{2}{3}$$

[B] $a_n = 2ne^{-n}$, $n \geq 1$

Sol: STEP 1: INCR/DECR:

$$f(n) = 2ne^{-n}$$

$$f'(n) = -2ne^{-n} + 2e^{-n}$$

$$= e^{-n}(2-2n) < 0 \text{ for } n \geq 1$$

for $n \geq 1$ $2-2n \leq 0$

So $\{a_n\}$ IS DECREASING.

STEP 2: CHECKING BOUNDEDNESS

* SINCE $\{a_n\}$ IS DECREASING. IT IS BOUNDED ABOVE BY a_1

$$M = 2e^{-1} = \boxed{2/e = M}$$

* TO SEE IF IT'S BOUNDED BELOW, TAKE LIMIT:

$$\lim_{n\to\infty} 2ne^{-n} = \lim_{n\to\infty}\left(\frac{2n}{e^n}\right)$$

L.R.

$$= \lim_{n\to\infty}\left(\frac{2}{e^n}\right) = \boxed{0. = m}$$

So $\{a_n\}$ IS BOUNDED

$$\boxed{0 \leq a_n \leq 2/e}$$