CH 11.10/11.11 TAYLOR SERIES and APPLICATIONS

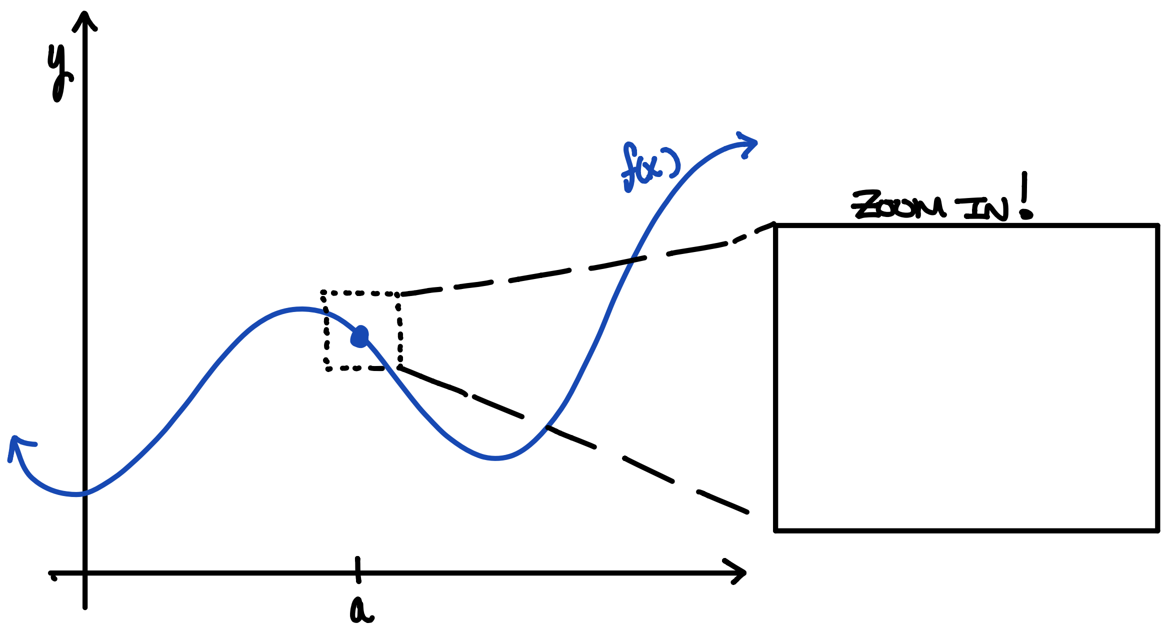
GOAL: In Ch 11.9 we were able to find POWER SERIES REPRESENTATIONS for functions that had the form $f(x) = \frac{1}{1-\square}$

Now, we will learn a technique to do this for more GENERAL functions! What we want is to be able to express a function as a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

PART 1: THE BASICS

How can we "approximate" a function f(x) near a point x=a with a polynomial?



LEVEL #1: DEGREE 1.

LEVEL #2: DEGREE 2

LEVEL #3: DEGREE 3

LEVEL #N:

THE **TAYLOR POLYNOMIAL** of DEGREE "N"

$f(x)$ is N-TIMES DIFFERENTIABLE NEAR $x=a$

$$T_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

$T_0(x)$ = DEG 0 TAYLOR POLYNOMIAL = KEEP ALL TERMS w/ DEGREE ≤ 0

$T_1(x)$ = DEG 1 TAYLOR POLYNOMIAL = KEEP ALL TERMS w/ DEGREE ≤ 1

$T_2(x)$ = DEG 2 TAYLOR POLYNOMIAL = KEEP ALL TERMS w/ DEGREE ≤ 2

$T_3(x)$ = DEG 3 TAYLOR POLYNOMIAL = KEEP ALL TERMS w/ DEGREE ≤ 3

If we let N go to infinity then we get a SERIES:

THE **TAYLOR SERIES** of $f(x)$ at $x=a$

$f(x)$ infinitely differentiable near a.

$$f(x) \underset{\text{ON I.O.C.}}{=} \text{Taylor Series} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

If the center "a" of the series is a=0, then we *sometimes* use another name for the series:

THE **MACLAURIN SERIES** of $f(x)$

$$f(x) \underset{\text{ON I.O.C.}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots$$

* WE WILL FIND THAT:

OUR **JOB**:

1. Find Taylor Series/Polynomials (and R.o.C's)
2. Verify that f(x) is EQUAL TO its Taylor series on the I.o.C.
3. Discuss ERROR associated with polynomial approximations
4. Understand why this is AWESOME!

PART 2: FINDING TAYLOR SERIES

* The first method is called: **BRUTE FORCE!**

Ex 1. Assuming (for now) that they exist, find the TAYLOR SERIES for each function centered at x=a (where a is specified) and find the associated radius of convergence.

$\boxed{A}$ $f(x) = e^x$ @ $x = 0$. $(a=0)$

Sol:

$\boxed{B}$ $f(x) = \sin(x)$ @ $x=0$

Sol:

NOTE: WE MAY JUST BE ASKED FOR A TAYLOR POLYNOMIAL... IN THIS CASE:

$\boxed{C}$ $f(x) = \ln(1+x)$ @ $x = 0$.

Sol:

* Some MACLAURIN SERIES for common functions can be seen in the following table. We found some of these above. Now we will use them to find more!

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

* Our second method of finding Taylor series involves: NEW from OLD

Ex 2: Using the table above, find the MACLAURIN SERIES for each of the following functions and find the associated radius of convergence.

$\boxed{A}$ $f(x) = e^{2x}$

sol:

$\boxed{C}$ $f(x) = x\sin(x)$

sol:

$\boxed{B}$ $f(x) = \cos(x^4)$

sol:

PART 3: WHEN DOES $f(x)$ EQUAL $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

* If a function is infinitely differentiable, when will the Taylor Series be EQUAL to the function? We must consider two things:
  - Find where the Taylor Series converges (I.e. the interval of convergence).
  - Verify that the partial sums of the series do in fact approach the function.

THEOREM: [TAYLOR SERIES vs. $f(x)$]

If $f(x) = T_N(x) + R_N(x)$, where $T_N(x)$ is the Nth degree Taylor Polynomial of f(x) centered at x=a and if for |x-a|<R

$$\lim_{N\to\infty} R_N(x) = 0$$

Then f(x) is EQUAL to the sum of its Taylor Series on the interval |x-a|<R.

NOTE: To show that $\lim_{N\to\infty} R_N(x) = 0$ we have a very useful inequality:

TAYLOR'S INEQUALITY

If $|f^{(N+1)}(x)| \le M$ for $|x-a| \le d$ then the remainder of the Nth degree Taylor Polynomial approximating f(x) at x=a satisfies:

$$|R_N(x)| \le \frac{M|x-a|^{N+1}}{(N+1)!} \quad \text{for} \quad |x-a| \le d$$

$$\text{for all } x \quad \lim_{n\to\infty}\left(\frac{x^n}{n!}\right) = 0$$

Ex 3. Prove that the MACLAURIN SERIES for sin(x) found in example 1 is equal to the function for all x.

sol:

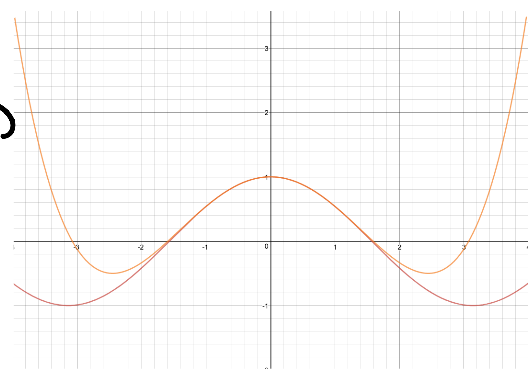
NOTE: TAYLOR POLYNOMIALS can work very well to approximate a function (provided you are on the interval of convergence). Of course, the larger N (i.e. the more terms we take in the polynomial), the better the approximation gets.

PART 4: ERROR IN TAYLOR POLYNOMIALS

** To determine how effective a Taylor Polynomial is at approximating a function, we look at the ERROR/REMAINDER.

3 WAYS

TO APPROXIMATE ERROR: $\boxed{|R_N(x)| = |f(x) - T_N(x)|}$ ON AN INTERVAL



① USE A GRAPH. GRAPH $f(x)$ & $T_N(x)$ AND LOOK FOR MAXIMUM GAP BETWEEN GRAPHS ON INTERVAL

② USE TAYLOR'S INEQUALITY.

$$|R_N(x)| \le \frac{M|x-a|^{N+1}}{(N+1)!}$$

③ USE ALTERNATING SERIES EST. THM. $\sum_{n=1}^{\infty} (-1)^n b_n$

$$|R_N(x)| \le b_{N+1}$$

Ex 5. For $f(x)=e^{2x^2}$, find the TAYLOR POLYNOMIAL of degree 3 centered at $a=0$ and then use TAYLOR'S INEQUALITY to estimate the error when $0\le x\le 0.1$

sol:

Ex 6. The 3rd degree TAYLOR POLYNOMIAL (Abt $x=0$) for $f(x)=\sin(x)$ is given by $T_3(x) = x - x^3/3!$
Use TAYLOR'S INEQUALITY to determine a "d" for which $|\sin(x)-T_3(x)| \leq 0.001$ for all x in [-d,d].

sol:

Ex 7. Use TAYLOR'S INEQUALITY to determine an N for which the Nth degree TAYLOR POLYNOMIAL for $f(x) = \sin(x)$ centered at $a=0$ satisfies $|\sin(3) - T_N(3)| \leq 0.0005$

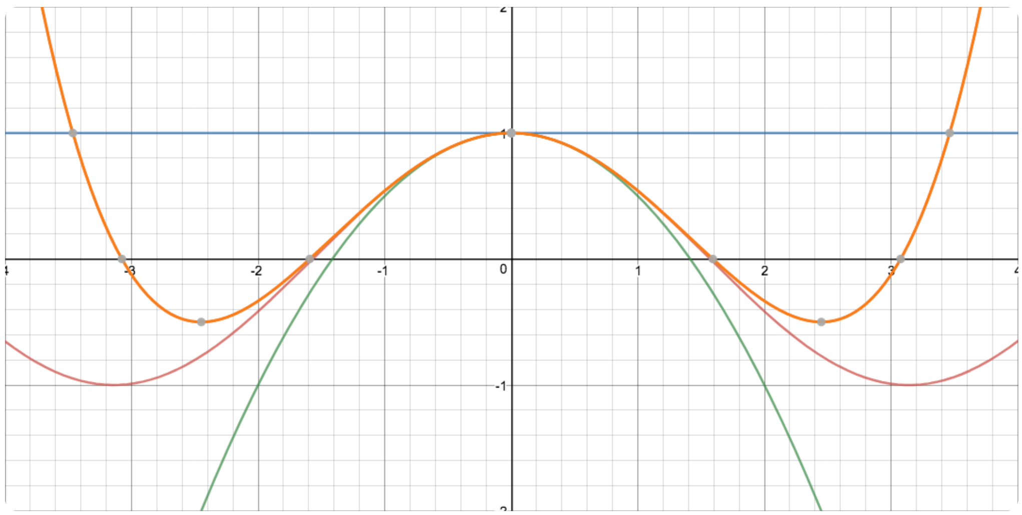
sol:

Ex 8. Use the ALTERNATING SERIES ESTIMATION THEOREM to estimate the range of values of x for which the approximation $\arctan(x) \approx x - x^3/3 + x^5/5$ is accurate to within 0.0002.

sol:

PART 3: WHY THIS IS AWESOME...

* Let's take a step back to see what all of this means and why we actually care about it!



A FUNCTION IS EQUAL TO ITS TAYLOR SERIES ON THE I.o.C. SO WE CAN USE THE TAYLOR POLYNOMIALS TO APPROXIMATE f(x).
WE LOVE POLYNOMIALS!

* One application of Taylor series is that they can now be used to APPROXIMATE Integrals!

Ex. Evaluate the integral as an infinite series. $\int \frac{e^{2x}}{x}\,dx.$