CH 11.10/11.11 TAYLOR SERIES and APPLICATIONS

GOAL: In Ch 11.9 we were able to find POWER SERIES REPRESENTATIONS for functions that had the form $f(x)=\frac{1}{1-\square}$

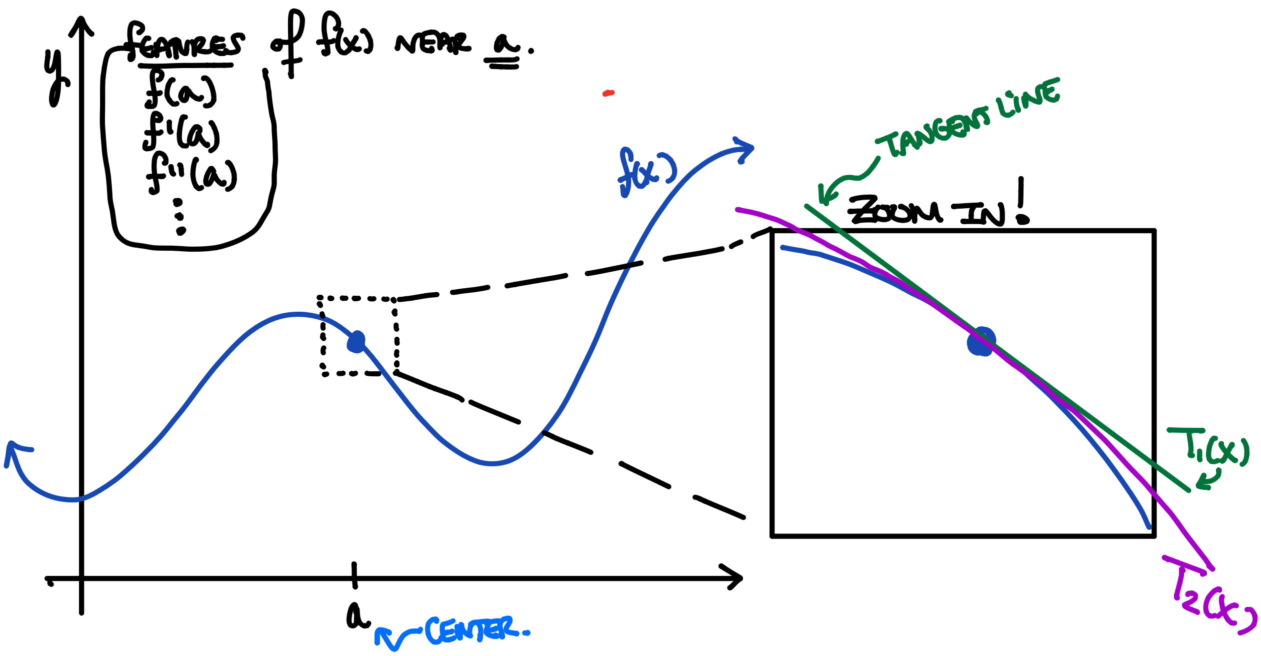
Now, we will learn a technique to do this for more GENERAL functions!
What we want is to be able to express a function as a power series:

$$\underbrace{f(x)}_{\text{function}} = \underbrace{\sum_{n=0}^{\infty} \overbrace{C_n}^{\text{constants}} (x-\overbrace{a}^{\text{center}})^n}_{\text{power series}} = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

POWER SERIES ~ Polynomial!

PART 1: THE BASICS

How can we "approximate" a function f(x) near a point x=a with a polynomial?



LEVEL #1: DEGREE 1. $T_1(x) = C_0 + C_1(x-a)$

MATCH:
VALUE $T_1(a) = f(a) \Rightarrow T_1(a) = C_0 = f(a)$ ✓
1st DER $T_1'(a) = f'(a) \Rightarrow C_1 = f'(a)$ ✓

$$T_1(x) = f(a) + f'(a)(x-a)$$

MATCHES VALUE & FIRST DER of f(x) @ a.

LEVEL #2: DEGREE 2

$$T_2(x) = C_0 + C_1(x-a) + C_2(x-a)^2$$

MATCH:

VALUE: $C_0 = f(a)$ $\quad T_2(a) = f(a)$

1st DER: $T_2'(a) = f'(a)$ $\quad C_1 = f'(a)$

2nd DER: $T_2''(a) = f''(a)$ $\quad 2C_2 = f''(a)$

$$\boxed{T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$

LEVEL #3: DEGREE 3

$$\boxed{T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$$

LEVEL #N:

THE **TAYLOR POLYNOMIAL** of DEGREE "N"

$f(x)$ is N-TIMES DIFFERENTIABLE NEAR $x=a$

$$T_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(N)}(a)}{N!}(x-a)^N.$$

$T_0(x)$ = DEG 0 TAYLOR POLYNOMIAL = KEEP ALL TERMS w/ DEGREE ≤ 0

$T_1(x)$ = DEG 1 TAYLOR POLYNOMIAL TANGENT LINE KEEP ALL TERMS w/ DEGREE ≤ 1

$T_2(x)$ = DEG 2 TAYLOR POLYNOMIAL = KEEP ALL TERMS w/ DEGREE ≤ 2

$T_3(x)$ = DEG 3 TAYLOR POLYNOMIAL = KEEP ALL TERMS w/ DEGREE ≤ 3

! Sometimes Taylor Polynomials CAN EQUAL EACH OTHER ($T_2(x) = T_3(x)$)

If we let N go to infinity then we get a SERIES:

THE TAYLOR SERIES of $f(x)$ at $x=a$

$f(x)$ infinitely differentiable near a.

$$f(x) \underset{\text{ON I.o.C.}}{=} \text{Taylor Series} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

CENTER.

If the center "a" of the series is a=0, then we *sometimes* use another name for the series:

THE MACLAURIN SERIES of $f(x)$

$$f(x) \underset{\text{ON I.o.C.}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots$$

$f(x)$ EQUALS Taylor Series ON I.o.C.

OUR JOB:

1. Find Taylor Series/Polynomials (and R.o.C's)
2. Understand why this is so AWESOME!
3. Discuss ERROR associated with polynomial approximations
4. Verify that f(x) is EQUAL TO its Taylor series on the I.o.C.
5. Look at some APPLICATINS

PART 2: FINDING TAYLOR SERIES

* The first method is called: **BRUTE FORCE!**

- Find $f(a), f'(a), f''(a)$, etc.
- Look for patterns.
- Input $a, f(a), f'(a), f''(a), \dots$ into formula.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

Ex 1. Assuming (for now) that they exist, find the TAYLOR SERIES for each function centered at x=a (where a is specified) and find the associated radius of convergence.

$\boxed{A}$ $f(x) = e^x$ @ $x=0$. $(a=0)$

Sol:

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = e^0 = 1$

$f''(x) = e^x \quad$ @ $x=0$ $\quad f''(0) = e^0 = 1$

$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$

$\vdots$

$$f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

$\boxed{a=0}$

$$1 + 1(x-0) + \frac{1(x-0)^2}{2!} + \frac{1(x-0)^3}{3!} \dots$$

Taylor series of e^x

$$e^x = \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots} \qquad \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

on $(-\infty, \infty)$

* To find R.o.C use RATIO TEST (CH 11.8)

$\boxed{R = \infty}$

$\boxed{B}$ $f(x) = \sin(x)$ @ $x=0$

Sol:

$f(x) = \sin(x) \qquad f(0) = 0$

$f'(x) = \cos(x) \qquad f'(0) = 1$

$f''(x) = -\sin(x) \quad$ @ $x=0$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x) \qquad f'''(0) = -1$

$f''''(x) = \sin(x) \qquad f''''(0) = 0$

$\vdots$

NOTE: We may just be asked for a TAYLOR POLYNOMIAL... IN CASE:

0 DEG	$T_0(x) = 0$
1 DEG	$T_1(x) = x$
2 DEG	$T_2(x) = x$
3 DEG	$T_3(x) = x - \frac{x^3}{3!}$
4 DEG	$T_4(x) = x - \frac{x^3}{3!}$

$$\text{TAYLOR SERIES} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 0 + 1(x) + 0 - \frac{1}{3!}(x)^3 \dots$$

$$f(x) = \sin(x) = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

on I.o.C $(-\infty, \infty)$ — Taylor series $\sin(x)$.

use RATIO TEST: R.o.C $= \infty$

$\boxed{C}$ $f(x) = \ln(1+x)$ @ $x = 0$.

Sol:

$$f(x) = \ln(1+x) \qquad f(0) = \ln(1) = 0$$

$$f'(x) = (1+x)^{-1} \qquad f'(0) = 1 = 0!$$

$$f''(x) = -(1+x)^{-2} \quad @\ x=0 \quad f''(0) = -1 = -1!$$

$$f'''(x) = 2(1+x)^{-3} \qquad f'''(0) = 2 = 2!$$

$$f''''(x) = -3 \cdot 2 (1+x)^{-4} \qquad f''''(0) = -3 \cdot 2 = -3!$$

$$f^{(5)}(x) = 4 \cdot 3 \cdot 2 (1+x)^{-5} \qquad f^{(5)}(0) = 4 \cdot 3 \cdot 2 = 4!$$

Taylor series $= f(a) + f'(a)(x-a) + \dfrac{f''(a)(x-a)^2}{2!} + \dfrac{f'''(a)(x-a)^3}{3!} + \dots$

$$= 0 + 1x + \frac{-1x^2}{2!} + \frac{2!x^3}{3!} - \frac{3!x^4}{4!} + \frac{4!x^5}{5!} - \dots$$

$$\ln(1+x) = \boxed{\underset{n=1}{x} - \underset{n=2}{\frac{x^2}{2}} + \underset{n=3}{\frac{x^3}{3}} - \underset{n=4}{\frac{x^4}{4}} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \dots}$$

on I.o.C.

$(-1, 1]$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$$

R.o.C = 1

* Some MACLAURIN SERIES for common functions can be seen in the following table. We found some of these above. Now we will use them to find more!

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

* Our second method of finding Taylor series involves: NEW from OLD

* OR FIND SIMPLE SERIES W/ NEW CENTER & THEN SUBSTITUTE.

! ONLY WORKS for MACLAURIN SERIES. If $a \neq 0$ USE BRUTE FORCE :)

Ex 2: Using the table above, find the MACLAURIN SERIES for each of the following functions and find the associated radius of convergence.

A $f(x) = e^{2x}$

! $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

sol: LET "x" $= 2x$

$$f(x) = e^{2x} = \boxed{\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}}$$

* R.o.C $= \infty$ (REMAINS SAME)

B $f(x) = \cos(x^4) =$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

sol:

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x^4)^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{8n}}{(2n)!}}$$

* R.o.C $= \infty$

C $f(x) = x\sin(x)$

Sol:

$$f(x) = x\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+1)!}} \quad R.o.C = \infty.$$

! $f(x) = \sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

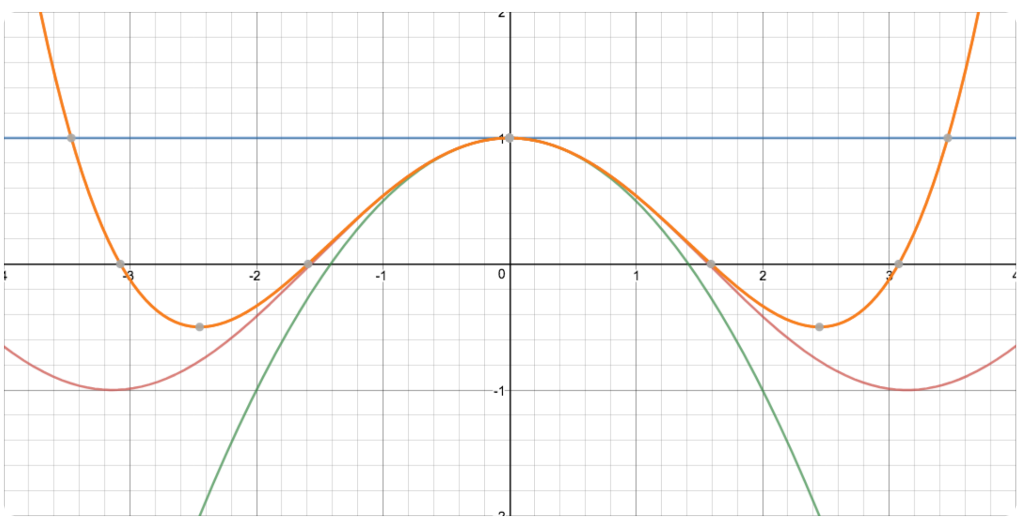
D $f(x) = x^4 e^{-x^2}$

Sol:

$$= x^4\left(\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}\right) = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n \cdot x^{2n+4}}{n!}} \quad R.o.C = \infty.$$

PART 3: WHY THIS IS AWESOME...

* Let's take a step back to see what all of this means and why we actually care about it!



! A FUNCTION IS EQUAL TO ITS TAYLOR SERIES ON THE I.o.C. SO WE CAN USE THE TAYLOR POLYNOMIALS TO APPROXIMATE $f(x)$.

WE LOVE POLYNOMIALS!

PART 4: WHEN DOES $f(x)$ EQUAL $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

* If a function is infinitely differentiable, when will the Taylor Series be EQUAL to the function? We must consider two things:
  - Find where the Taylor Series converges (I.e. the interval of convergence).
  - Verify that the $\underbrace{\text{partial sums}}_{T_N(x)\text{ Taylor Polynomial}}$ of the series do in fact approach the function.

GOAL: $f(x) = \lim_{N\to\infty}(T_N(x)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ on I.o.C.

$\boxed{\text{ERROR}}$ $R_N(x) = |f(x) - T_N(x)|$

SHOW: $R_N(x) \to 0$ as $N \to \infty$

THEOREM: [TAYLOR SERIES vs. $f(x)$]

If $f(x) = T_N(x) + R_N(x)$, where $T_N(x)$ is the Nth degree Taylor Polynomial of f(x) centered at x=a and if for $\underbrace{|x-a|<R}_{\text{I.o.C.}}$

$$\lim_{N\to\infty} R_N(x) = 0$$

Then f(x) is EQUAL to the sum of its Taylor Series on the interval $|x-a|<R$.

NOTE: To show that $\lim_{N\to\infty} R_N(x) = 0$ we have a very useful inequality:

TAYLOR'S INEQUALITY

If $|f^{(N+1)}(x)| \le M$ for $|x-a| \le d$ then the remainder of the Nth degree Taylor Polynomial approximating f(x) at x=a satisfies:

$$|R_N(x)| \le \underbrace{\frac{M|x-a|^{N+1}}{(N+1)!}}_{\text{BOUND ON ERROR.}} \quad \text{for } |x-a| \le d$$

$$\boxed{\text{for all } x \quad \lim_{n\to\infty}\left(\frac{x^n}{n!}\right)=0}$$

Ex 3. Prove that the MACLAURIN SERIES for sin(x) found in example 1 is equal to the function for all x.

sol: $f(x)=\sin(x) \overset{\text{show}}{=} \text{Taylor series} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$

Show $R_N(x) \longrightarrow 0$.

$$|R_N(x)| \le \frac{M|x-a|^{N+1}}{(N+1)!}$$

$$= \frac{1\,|x|^{N+1}}{(N+1)!}$$

find "M" on I.o.C $=(-\infty,\infty)$

$f(x)=\sin(x)$

$f'(x)=\cos(x)$

$f''(x)=-\sin(x)$

$f'''(x)=-\cos(x)$

$f^{(4)}(x)=\sin(x)$

$f^{(N+1)}(x)=\pm\sin(x)$ or $\pm\cos(x)$.

$|f^{(N+1)}(x)|=|\cos(x)|$ or $|\sin(x)|$

MAX on $(-\infty,\infty)$ of $\boxed{1=M}$

$$\lim_{N\to\infty} 0 \le \lim_{N\to\infty}|R_N(x)| \le \lim_{N\to\infty}\frac{|x|^{N+1}}{(N+1)!}$$

$$0 \le \lim_{N\to\infty}|R_N| \le 0$$

SQUEEZE THM. $\boxed{R_N \to 0}$

So $\boxed{\sin(x) \overset{\text{Always.}}{=} x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots}$

NOTE: **TAYLOR POLYNOMIALS** can work very well to approximate a function (provided you are on the interval of convergence). Of course, the larger N (i.e. the more terms we take in the polynomial), the better the approximation gets.

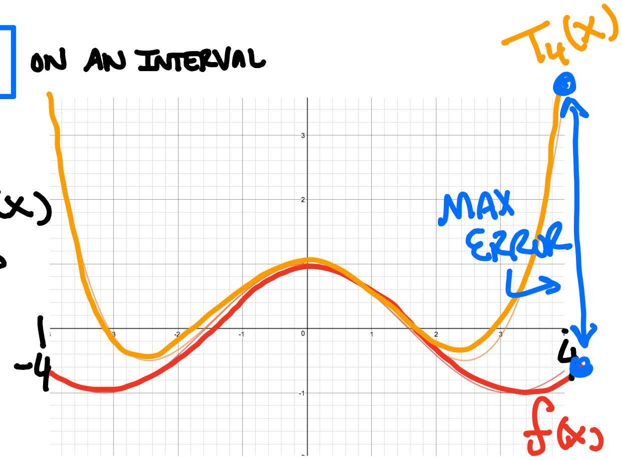
PART 5: ERROR IN TAYLOR POLYNOMIALS

** To determine how effective a Taylor Polynomial is at approximating a function, we look at the ERROR/REMAINDER.

$$|R_N(x)| = |f(x) - T_N(x)|.$$

3 WAYS

TO APPROXIMATE ERROR: $\boxed{|R_N(x)| = |f(x) - T_N(x)|}$ ON AN INTERVAL



① USE A GRAPH. GRAPH $f(x)$ & $T_N(x)$ AND LOOK FOR MAXIMUM GAP BETWEEN GRAPHS ON INTERVAL

② USE TAYLOR'S INEQUALITY.

$$|R_N(x)| \le \frac{M|x-a|^{N+1}}{(N+1)!}$$

③ USE ALTERNATING SERIES EST THM. $\sum (-1)^n \cdot b_n$

$$|R_N(x)| \le b_{N+1}$$

Ex 5. For $f(x)=e^{2x^2}$, find the **TAYLOR POLYNOMIAL** $T_3(x)$ of degree 3 centered at $a=0$ and then use **TAYLOR'S INEQUALITY** to estimate the error when $0 \le x \le 0.1$

sol: $\boxed{T_3(x) = 1+2x^2}$ vs. $f(x) = e^{2x^2}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{2x^2} = \underbrace{1 + (2x^2)}_{T_3(x)} + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \dots$$

$$|R_N(x)| \le \frac{M|x|^{N+1}}{(N+1)!} = 52.9\frac{|x|^4}{4!}$$

WORST CASE

$$\le \frac{52.9(0.1)^4}{4!}$$

$$= \boxed{0.00022}$$

MAX ERROR.

FIND M: $M \ge |f^{(4)}(x)|$

$f(x) = e^{2x^2}$

$\vdots$

$f''''(x) = 256x^4e^{2x^2} + 384x^2e^{2x^2} + 48e^{2x^2}$

MAX ON $[0, 0.1]$ @ 0.1

$\boxed{M = 52.9}$

Ex 6. The 3rd degree TAYLOR POLYNOMIAL (Abt $x=0$) for $f(x)=\sin(x)$ is given by $T_3(x)=x-x^3/3!$
Use TAYLOR'S INEQUALITY to determine a "d" for which $|\underbrace{\sin(x)-T_3(x)}_{\text{ERROR}}| \le \underbrace{0.001}_{\text{ACCURACY}}$ for all x in [-d,d].

Sol:

$$f(x)=\sin(x) \quad \text{vs.} \quad T_3(x)=x-\frac{x^3}{3!}$$

$$|R_N(x)| \le \frac{M|x-a|^{N+1}}{(N+1)!} \le 0.001$$

$$\frac{M\,|x|^4}{4!} \le 0.001$$

$$\frac{|x|^4}{4!} \le 0.001$$

Solve for x.

$$|x| \le \sqrt[4]{0.024} = 0.394$$

$$-0.394 \le x \le 0.394 \qquad d=0.394$$

FIND M

$f(x)=\sin(x)$

$|f^{(N+1)}| = |\sin(x) \text{ OR } \cos(x)|$

$M=1$

Ex 7. Use TAYLOR'S INEQUALITY to determine an N for which the Nth degree TAYLOR POLYNOMIAL for $f(x)=\sin(x)$ centered at $a=0$ satisfies $\underbrace{|\sin(3)-T_N(3)|}_{x=3} \leq \underbrace{0.0005}_{\text{ACCURACY}}$

sol: $f(x)=\sin(x)$ vs. $T_N(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+$

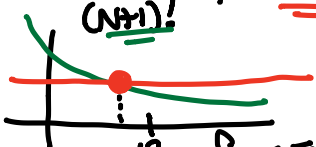
$$|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!} \leq 0.0005$$

$$\frac{1\,|x|^{N+1}}{(N+1)!} \leq 0.0005$$

SOLVE FOR N $\quad \frac{3^{N+1}}{(N+1)!} \leq 0.0005$

DESMOS:

GRAPH $\frac{3^{N+1}}{(N+1)!}$ & 0.0005



WE NEED $N=12$.

$$\boxed{T_{12}(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\cdots-\frac{x^{11}}{11!}}$$

Ex 8. Use the ALTERNATING SERIES ESTIMATION THEOREM to estimate the range of values of x for which the approximation $\arctan(x)\approx x-x^3/3+x^5/5$ is accurate to within 0.0002. ← ACCURACY.

sol: $f(x)=\arctan(x)$ $\quad T_5(x)=x-\frac{x^3}{3}+\frac{x^5}{5}\underbrace{-\frac{x^7}{7}}_{\text{GHOST.}}$ ← MISSING.

$$|R_5(x)| \leq \left|\frac{x^7}{7}\right| \leq 0.0002$$

SOLVE FOR x.

$$|x| \leq \sqrt[7]{0.0014} = 0.391$$

$$\boxed{-0.391 \leq x \leq 0.391}$$

PART 6 : USING TAYLOR SERIES

* One application of Taylor series is that they can now be used to APPROXIMATE Integrals!

Ex 9. Evaluate the integral as an infinite series

$$\int \frac{e^{2x}}{3x}\,dx$$

Sol: STEP 1: Find Series $f(x) = \frac{e^{2x}}{3x}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$\frac{e^{2x}}{3x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!(3x)} = \sum_{n=0}^{\infty} \frac{2^n x^{n-1}}{n! \cdot 3}$$

STEP 2: INTEGRATE:

$$\int \sum_{n=0}^{\infty} \frac{2^n x^{n-1}}{n! \cdot 3}\,dx$$

$$= \sum_{n=0}^{\infty} \int \frac{2^n x^{n-1}}{n! \cdot 3}\,dx = \sum_{n=0}^{\infty} \frac{2^n}{n! \cdot 3} \int x^{n-1}\,dx$$

$n=0$ $n \geq 1$

$$\frac{1}{3}\int x^{-1}\,dx + \sum_{n=1}^{\infty} \frac{2^n}{n! \cdot 3} \int x^{n-1}\,dx = \frac{1}{3}\ln|x| + \sum_{n=1}^{\infty} \frac{2^n x^n}{n! \cdot (3n)} + C$$