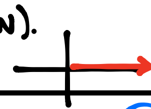
CH 10.4. AREA IN POLAR

GOAL: We will take a closer look at graphing equations using polar coordinates and we will then investigate how to find the AREA enclosed by a polar graph.

! New terminology:
The POLE: $\longrightarrow (0,0)$ (i.e. ORIGIN).
The POLAR AXIS: $\longrightarrow$ Pos. X Axis 

PART 1: CONVERTING CARTESIAN EQUATIONS $\Leftrightarrow$ POLAR EQUATIONS

** Sometimes it can be beneficial to convert between equations given in CARTESIAN coordinates and equations given in POLAR equations. To do this, we use what we know about converting coordinates...

Ex 1. Convert the following equation (given in Cartesian coordinates) to an equation given in Polar coordinates:

$$2(x^2+y^2)=4y$$

sol:

$$\boxed{\begin{aligned} r^2&=x^2+y^2\\ x&=r\cos\theta\\ y&=r\sin\theta \end{aligned}}$$

$$2\cdot r^2 = 4r\sin\theta$$

$r \neq 0$

$$\boxed{r = 2\sin\theta}$$

Ex 2. Convert the following equation (given in Polar coordinates) to an equation given in Cartesian coordinates:

$$r = 2\cos(\theta).$$

sol:

$$\boxed{\begin{aligned} r^2&=x^2+y^2\\ x&=r\cos\theta\\ y&=r\sin\theta \end{aligned}}$$

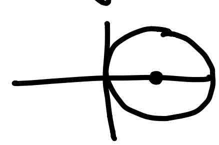
$$r = 2\cdot\frac{x}{r}$$

$$r^2 = 2x$$

$$\boxed{x^2+y^2 = 2x}$$

$$x^2-2x+y^2=0$$

$$(x-1)^2+y^2=1$$



PART 2: FINDING INTERSECTION of 2 POLAR CURVES.

** It is often necessary to find the intersection points of two curves given in polar coordinates. This can be TRICKY since points (and curves) can have multiple representations in polar.

To find all intersection points:

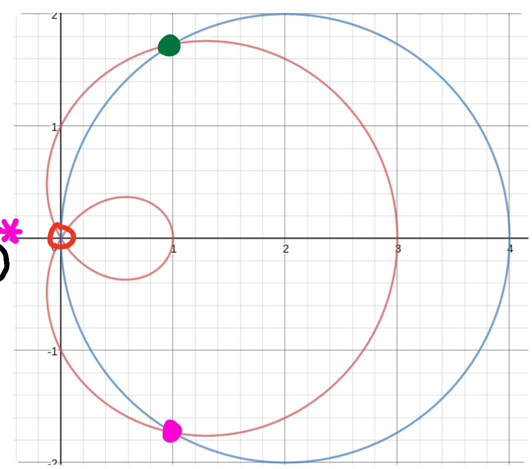
- STEP 1: SET EQUATIONS EQUAL. SOLVE FOR θ ← θ's WHERE CURVES INTERSECT.
  FIND CORRESPONDING r's.
- STEP 2: CHECK ON GRAPH TO MAKE SURE YOU DIDN'T MISS ANY.

Ex 3. Find all intersection points of the following curves given in polar coordinates. Verify the intersection points graphically!

A $r = 1 + 2\cos(\theta)$ and $r = 4\cos(\theta)$

sol:
$$1 + 2\cos\theta = 4\cos\theta$$
$$\cos\theta = \tfrac{1}{2} \Rightarrow \theta = \pi/3,\ 5\pi/3, \ldots$$
$$\theta = \pi/3 \Rightarrow r = 4\cos(\pi/3) = 2 \qquad (2, \pi/3)$$
$$\theta = 5\pi/3 \Rightarrow r = 4\cos(5\pi/3) = 2 \qquad (2, 5\pi/3)$$

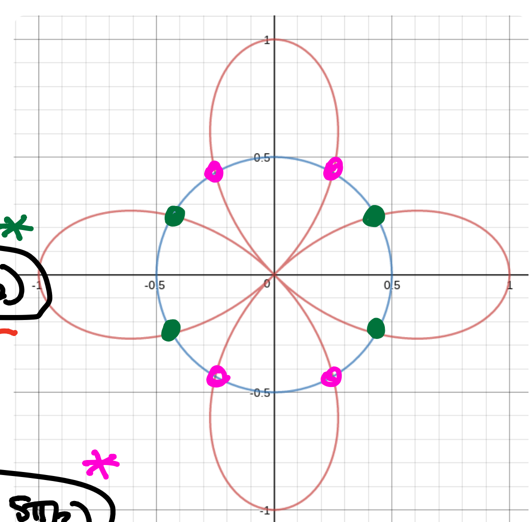
! $(0,0)$



B $r = \cos(2\theta)$ & $r = 1/2$ — $r = -1/2$

sol:
$$\cos(2\theta) = 1/2$$
$$2\theta = \pi/3,\ 5\pi/3,\ 7\pi/3,\ 11\pi/3, \ldots$$
$$\theta = \pi/6,\ 5\pi/6,\ 7\pi/6,\ 11\pi/6, \ldots$$
$$\boxed{(1/2, \pi/6),\ (1/2, 5\pi/6),\ (1/2, 7\pi/6),\ (1/2, 11\pi/6)}$$

$$\cos\theta = -1/2$$
$$2\theta = 2\pi/3,\ 4\pi/3,\ 8\pi/3,\ 10\pi/3, \ldots$$
$$\pi/3,\ 2\pi/3,\ 4\pi/3,\ 5\pi/3$$
$$\boxed{(-1/2, \pi/3),\ (-1/2, 2\pi/3),\ (-1/2, 4\pi/3),\ (-1/2, 5\pi/3)}$$

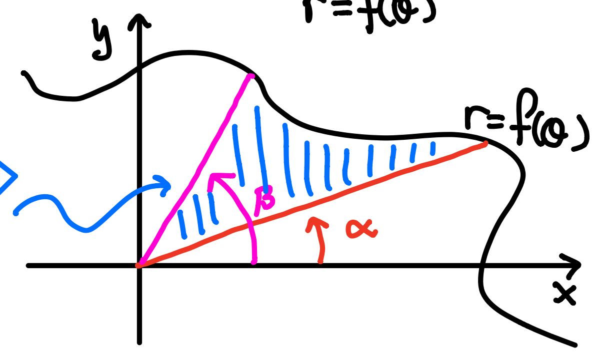


PART 3: AREA IN POLAR

****** We will derive a formula that can be used to find the **AREA** enclosed by a polar curve.

formula for AREA

AREA ENCLOSED BTW $\theta = \alpha$ & $\theta = \beta$ $= \displaystyle\int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2\, d\theta.$



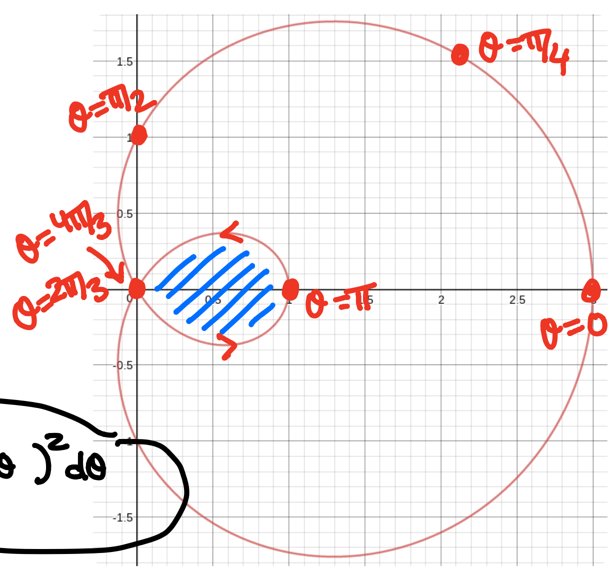
TRICKY

To determine the <u>bounds</u> of integration, you MUST make sure that you know how the polar curve is drawn for increasing theta. It can be deceptive! Consider this example:

$r = 1 + 2\cos(\theta)$.

θ	r
0	3
$\pi/4$	$1+\sqrt{2}$
$\pi/2$	1
$2\pi/3$	0
π	-1
$4\pi/3$	0

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(1+2\cos\theta)^2\, d\theta$$



** Let's check out how to find area of circles:

Ex. Find the **AREA** of each polar region:

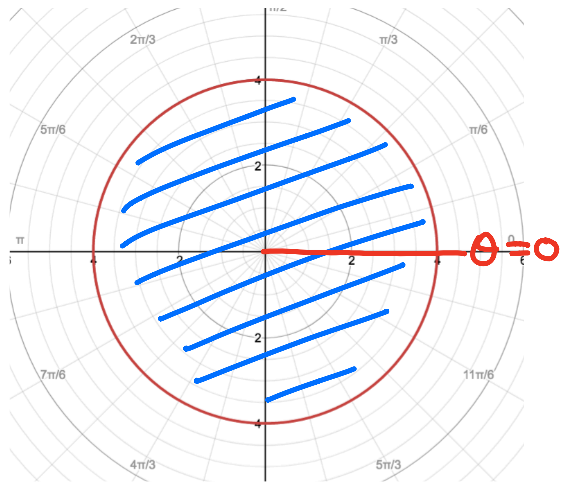
A $r = 4$

$f(\theta)$

$$A = \int_{\alpha = 0}^{\beta = 2\pi} \frac{1}{2}\,(f(\theta))^2\, d\theta$$

$$\int_0^{2\pi} \frac{1}{2}(4)^2\, d\theta = 8\theta \Big|_0^{2\pi}$$

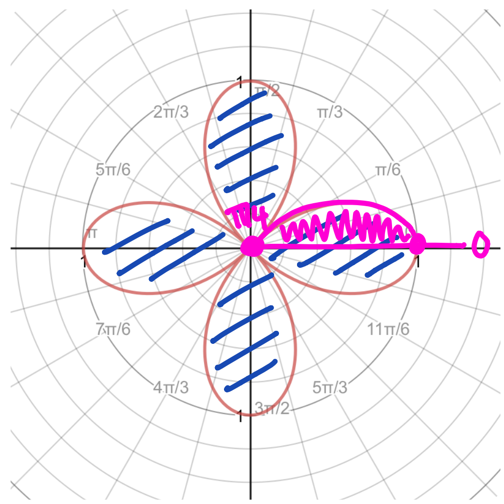
$$= \boxed{16\pi}$$



B $r = \cos(2\theta)$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\,(f(\theta))^2\, d\theta$$

$$= 8\int_0^{\pi/4} \frac{1}{2}(\cos(2\theta))^2\, d\theta$$



BE CAREFUL!

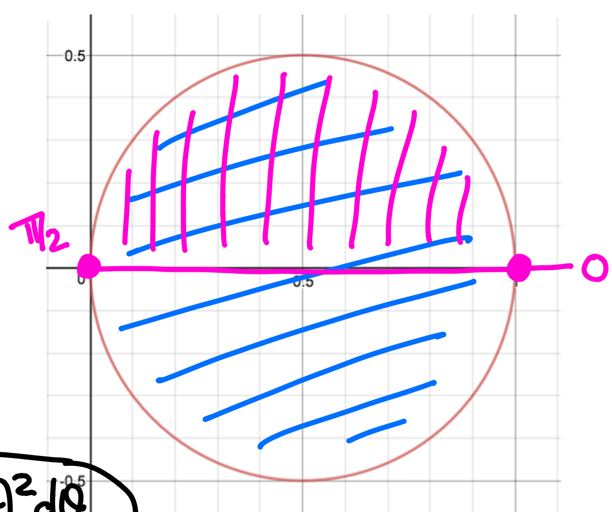
C $r = \cos(\theta)$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\,(f(\theta))^2\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2}(\cos\theta)^2\, d\theta.$$

OR

$$\int^{\pi} \frac{1}{2}(\cos\theta)^2\, d\theta$$



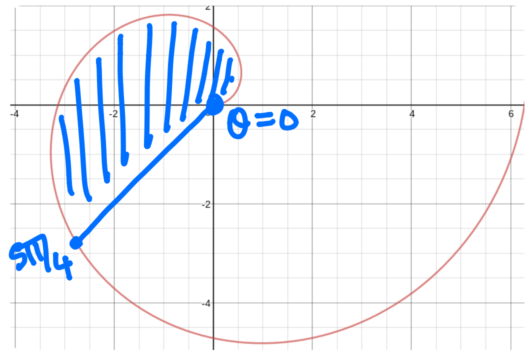
Ex 5. Find the AREA of each of the following regions:

A $r=\theta$

Sol:

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2\, d\theta$$

$$= \int_0^{5\pi/4} \frac{1}{2}(\theta)^2\, d\theta$$



B $r=\sin(4\theta)$ 1 Petal

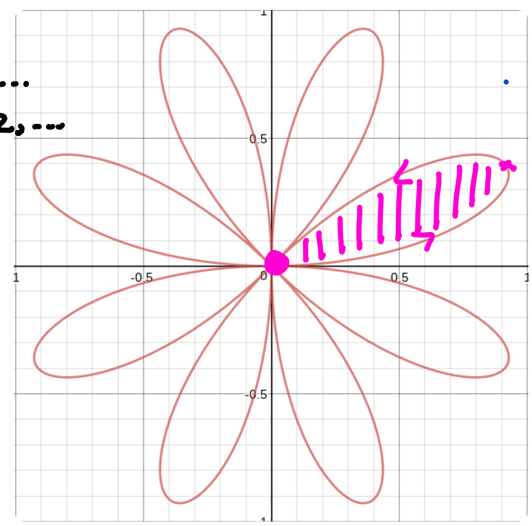
Sol:

$$\int_0^{\pi/4} \frac{1}{2}(\sin(4\theta))^2\, d\theta$$

⚠ $\sin(4\theta)=0$

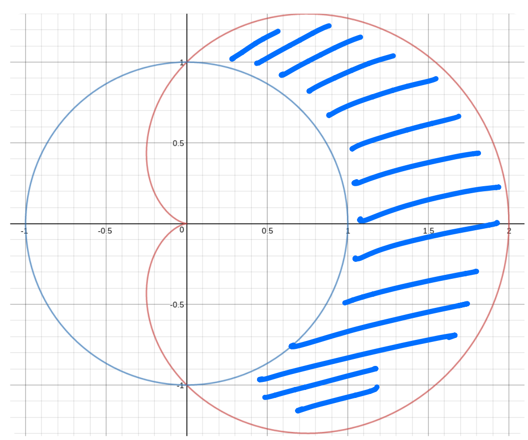
$4\theta = 0, \pi, 2\pi, \ldots$

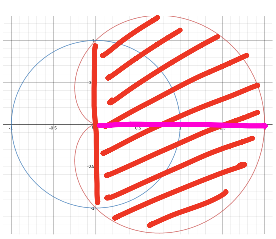
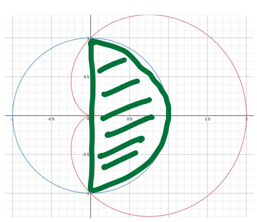
$\theta = 0, \pi/4, \pi/2, \ldots$



C $r=1-\cos\theta$; $r=1$

Sol:



$=$  $-$ 

$$= \cancel{2}\int_0^{\pi/2} \frac{1}{\cancel{2}}(1-\cos\theta)^2\, d\theta - \cancel{2}\int_0^{\pi/2} \frac{1}{\cancel{2}}(1)^2\, d\theta$$

$$= \int_0^{\pi/2} \left((1-\cos\theta)^2 - 1\right) d\theta$$

OR $$\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[(1-\cos\theta)^2 - 1\right] d\theta$$