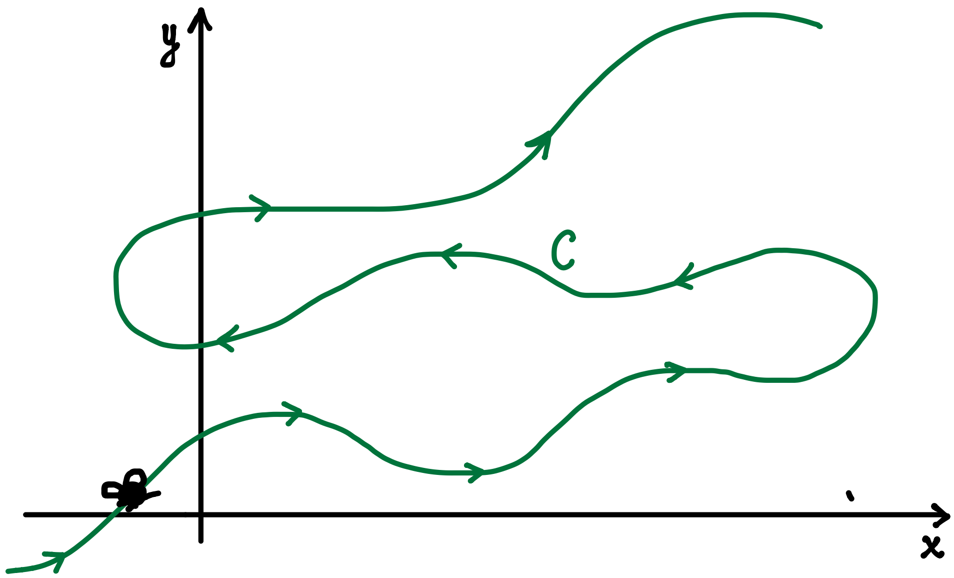
CH 10.1 PARAMETRIC EQUATIONS

MOTIVATION



** Consider a particle moving along the curve C in the indicated direction. We can't describe C as a function y=f(x) since the curve does not pass the vertical line test.... But we can still describe the motion of the particle using what we call **PARAMETRIC EQUATIONS.**

PART 1: THE BASICS

PARAMETER.

- Suppose x & y are fn's of "t"

$$x = f(t) \qquad y = g(t)$$

- for every t you get a point on the curve $(x(t), y(t))$

- Take all t's and you get a PARAMETRIC CURVE.

if $a \le t \le b$

$(x(a), y(a))$ = initial pt

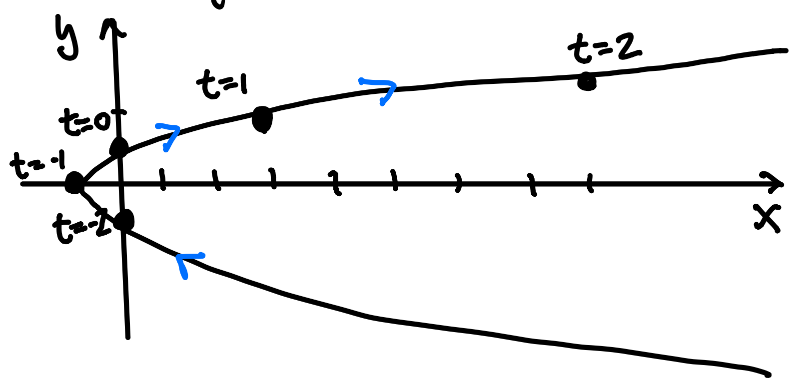
$(x(b), y(b))$ = final pt.

Ex 1: Sketch the curve defined by the **PARAMETRIC EQUATIONS**:

$$x = t^2 + 2t \quad \text{and} \quad y = t + 1$$

Sol:

t	x	y
-2	0	-1
-1	-1	0
0	0	1
1	3	2
2	8	3



SOMETIMES WE CAN ELIMINATE "t" & GET y AS A fn of x (OR VICE VERSA).

$$x = t^2 + 2t \quad \text{and} \quad y = t + 1$$

$$t = y - 1$$

$$x = (y-1)^2 + 2(y-1)$$

$$x = y^2 - 1$$

DESMOS DEMO: Let's look at some awesome curves that are defined parametrically.

Ex 1:

$x = t + \sin(5t)$
$y = t + \sin(6t)$

Ex 2:

$x = (\sin(7\pi t))^3$
$y = (\cos(5\pi t))^3$

Ex 3:

$x = 2.3\cos(10t) + \cos(23t)$
$y = 2.3\sin(10t) - \sin(23t)$

Ex 2. Consider the following PARAMETRIC CURVE:

$$(x,y) = (t-1,\ 4t) \qquad 0 \le t \le 2$$

!

$x = t-1$
$y = 4t$

A Write x in terms of y (I.e. eliminate the parameter t)

Sol:

$x = t-1$

Solve for t

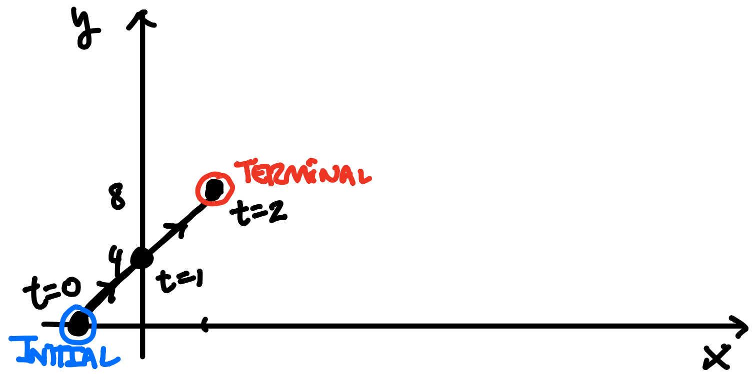
$t = x+1$ $\xrightarrow{\text{INPUT TO } y}$ $y = 4(x+1)$

$$\boxed{y = 4x+4}$$

B Draw a picture of the PARAMETRIC CURVE and clearly indicate the initial and terminal points as well as the direction of increasing values of "t".

Sol:

t	x	y
0	-1	0
1	0	4
2	1	8



Ex 3. Consider the following PARAMETRIC CURVE:

$$(x,y) = (t^2,\ 2t^3) \qquad 0 \le t \le 1$$

A Write x in terms of y (I.e. eliminate the parameter t)

Sol:

$x = t^2 \qquad y = 2t^3 \Rightarrow t = \sqrt[3]{y/2}$

$$\boxed{x = (y/2)^{2/3}}$$

B Draw a picture of the PARAMETRIC CURVE and clearly indicate the initial and terminal points as well as the direction of increasing values of "t".

Sol:

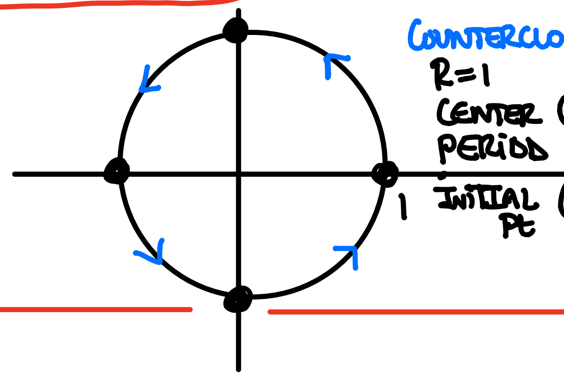
PART 2: ROTATIONAL PARAMETRIC CURVES "CIRCLES"

THE UNIT CIRCLE

* We can express the unit circle using parametric equations!

$$(x, y) = (\cos(t), \sin(t)) \quad 0 \le t \le 2\pi$$

t	x	y
0	1	0
$\pi/2$	0	1
π	-1	0
$3\pi/2$	0	-1
2π	1	0



COUNTERCLOCKWISE
$R=1$
CENTER $(0,0)$
PERIOD 2π
INITIAL Pt $(1,0)$

TRIG IDENTITIES

$\cos(-t) = \cos(t)$

$\sin(-t) = -\sin(t)$

$\cos(t + \pi/2) = -\sin(t)$

$\cos(t + \pi) = -\cos(t)$

$\sin(t + \pi/2) = \cos(t)$

$\sin(t + \pi) = -\sin(t)$

NOTE: Using the parametric representation of the unit circle... We can find parametric equations for circles with other properties!

UNIT CIRCLE

$(\cos(t), \sin(t))$

$0 \le t \le 2\pi$

① CENTER (h,k) RADIUS "r"

$(h + r\cos(t), k + r\sin(t))$.

$0 \le t \le 2\pi$

② CHANGE INITIAL Pt.

@ $t=0$

$(\cos(t+\theta), \sin(t+\theta))$

$0 \le t \le 2\pi$

STARTS @ θ.

③ REVERSE DIRECTION

REPLACE t w/ $-t$.

$\underbrace{(\cos(-t), \sin(-t))}_{\text{"CLOCKWISE"}} = \boxed{(\cos(t), -\sin(t))}$

Identities.

FASTER ROTATION

$(\cos(bt), \sin(bt))$

PERIOD $= \dfrac{2\pi}{b}$

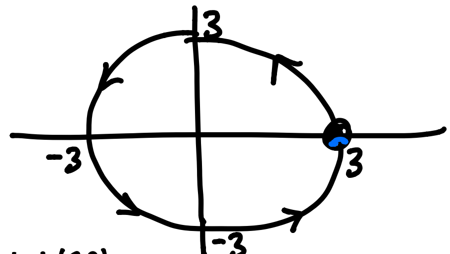
Ex 4. Find a PARAMETRIZATION of a circle centered at the origin with radius 3 that satisfies the following:

$(3\cos(t), 3\sin(t))$

- Counterclockwise orientation with initial point (3,0)

sol: ✓FREE ✓FREE.

$\boxed{(x,y) = (3\cos(t), 3\sin(t))}$



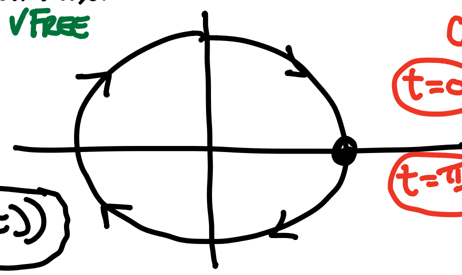
- Clockwise orientation with initial point (3,0)

sol: ✓FREE

CHANGE R — $(3\cos(t), 3\sin(t))$

CHANGE DIR. — $(3\cos(-t), 3\sin(-t))$

$= \boxed{(3\cos(t), -3\sin(t))}$



CHECK:

$t=0$ $(3\cdot 1, 0)$ ✓

$t=\frac{\pi}{2}$ $(0,-3)$ ✓

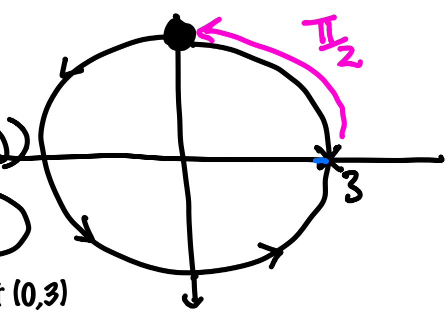
- Counterclockwise orientation with initial point (0,3)

sol: ✓FREE

CHANGE R — $(3\cos(t), 3\sin(t))$

CHANGE INT PT $(3\cos(t+\pi/2), 3\sin(t+\pi/2))$

$\boxed{(-3\sin(t), 3\cos(t))}$



- Clockwise orientation with initial point (0,3)

sol:

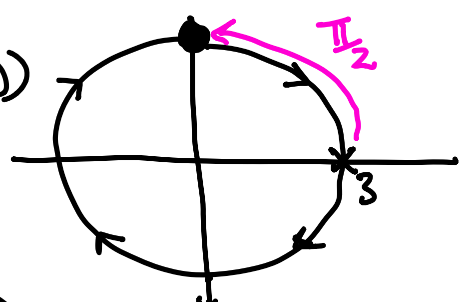
CHANGE R — $(3\cos(t), 3\sin(t))$

CHANGE INT PT $(3\cos(t+\pi/2), 3\sin(t+\pi/2))$

$(-3\sin(t), 3\cos(t))$

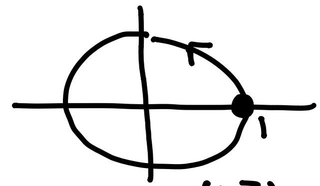
CHANGE DIR. $(-3\sin(-t), 3\cos(-t))$

$\boxed{(3\sin(t), 3\cos(t))}$



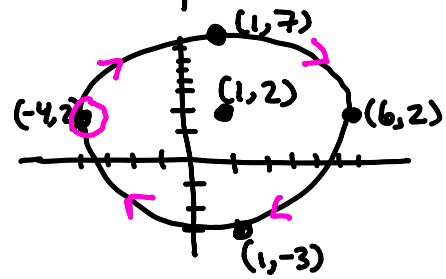
Ex 5. Find the PARAMETRIZATION of a circle with center (1,2), radius 5, oriented clockwise, with initial point (-4,2).

sol:



$(\cos(t), \sin(t))$

CHANGE R & CENTER $\quad (1+5\cos(t), 2+5\sin(t))$



CHANGE int pt $\quad (1+5\cos(t+\pi), 2+5\sin(t+\pi))$

$(1-5\cos(t), 2-5\sin(t))$

CHANGE DIR $\quad (1-5\cos(-t), 2-5\sin(-t))$

$$\boxed{(1-5\cos(t), 2+5\sin(t))}$$

CIRCLE: $(x-h)^2+(y-k)^2=R^2$

Ex 6. A fly moves along a circle $(x-1)^2+y^2=25$
Find a PARAMETRIC CURVE that describes that path of the fly in each of the following cases:

- The fly travels in a clockwise direction starting at (5,0) with

sol:

- The fly travels halfway around the circle in a counterclockwise direction starting at (1,5) with $0 \le t \le \pi$

sol:

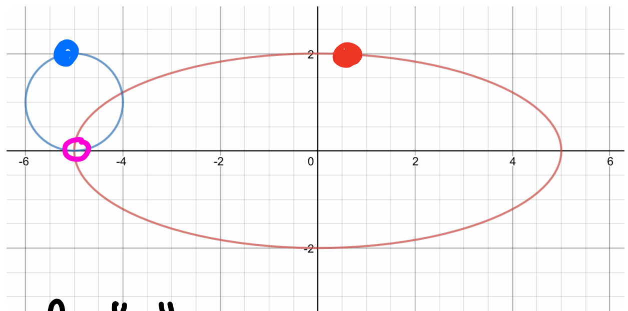
Ex 7. Two particles travel in space according the following **PARAMETRIC EQUATIONS:**

$$(x_1,y_1) = (5\sin(t), 2\cos(t)) \qquad 0 \le t \le 2\pi$$
$$(x_2,y_2) = (-5+\cos(t), 1+\sin(t))$$

Find ~~all~~ # points of intersection of the two paths and then find the location of any **COLLISION POINTS** (if they exist).

Sol: P.O.I.: 2 pts of INTERSECTION.

Collission Pt: Same Place @ Same time.



Set $x_1 = x_2$ $\quad y_1 = y_2$ $\quad$ Solve for "t"

$$5\sin(t) = -5+\cos(t) \qquad 2\cos(t) = 1+\sin(t)$$

$$\begin{array}{rl} & 5 = \cos(t) - 5\sin(t) \\ + & (1 = 2\cos(t) - \sin(t))(-5) \\ \hline & 0 = -9\cos(t) \Rightarrow \cos(t) = 0 \end{array}$$

$$\underbrace{t = \tfrac{\pi}{2}, \tfrac{3\pi}{2}}_{\text{Poss. Locations.}}$$

CHECK:

$\boxed{\cancel{t = \pi/2}}$ $(x_1,y_1) = (5,0)$ $\quad (x_2,y_2) = (-5,2)$

✓ $\boxed{t = 3\pi/2}$ $(x_1,y_1) = (-5,0)$ $\quad (x_2,y_2) = (-5,0)$

$$\boxed{\text{Coll. Pt of } (-5,0) \text{ @ } t = 3\pi/2}$$