CH 10.2 CALCULUS w/ PARAMETRIC CURVES

GOAL: We will learn how to find derivatives of parametric curves in order to find TANGENT LINES to the curves.

PART 1: SLOPE of A PARAMETRIC CURVE

! $dy/dx =$ SLOPE

Consider the parametric curve: $x=f(t),\ y=g(t)$
Where y is also a differentiable function of x. Notice:

CHAIN RULE $\Rightarrow$ $dy/dt = dy/dx \cdot dx/dt$ so $\boxed{dy/dx = \dfrac{dy/dt}{dx/dt}}$ SLOPE of PARAMETRIC CURVE.

* This allows us to the the SLOPE of the TANGENT LINE to a parametric curve at a given point without ever having to eliminate the parameter "t".

NOTE: HOR. TANGENT $\Rightarrow$ $dy/dt = 0$
VER. TANGENT $\Rightarrow$ $dx/dt = 0$

THE EQUATION of A TANGENT LINE:

$y - y_1 = m(x - x_1)$

m: dy/dx @ TIME t

! You may be asked to find slope at a point (x,y) OR at a time "t". You will need time "t" to find slope so if you are just given a point, then use x=f(t) or y=g(t) to find "t"

PART 2 EXAMPLES

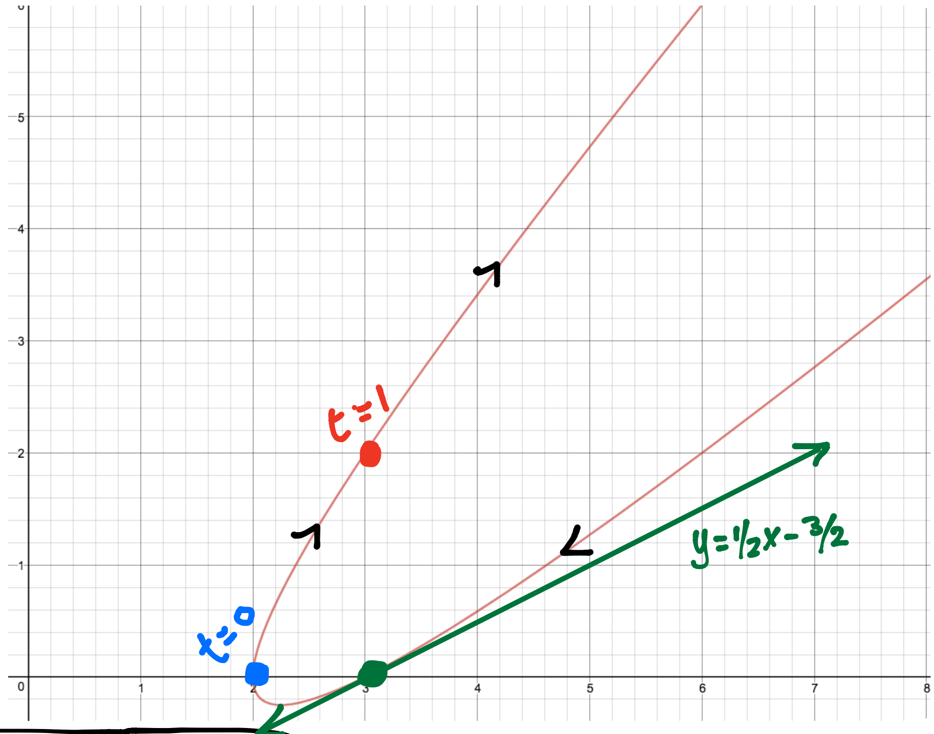
Ex 1. Consider the PARAMETRIC CURVE:

$$(x,y) = (2+t^2,\ t+t^2)$$

- Find the direction of increasing "t" values
- Find dy/dx in terms of "t"
- Find the equation of the TANGENT LINE to the curve at the point (3,0).
- Find the coordinates of points on the curve where the tangent is horizontal and vertical.

t	x	y
0	2	0
1	3	2

Sol:



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+2t}{2t}$$

$(x,y) = (2+t^2,\ t+t^2)$

find t w/ $(x,y)=(3,0)$
$x = 2+t^2 = 3$
$t^2 = 1 \Rightarrow t = 1$ OR -1
$y = t+t^2 = 0$
so $t=-1$

TANGENT @ (3,0)

POINT = (3,0) SLOPE = $dy/dx = \frac{1+2t}{2t}$ @ $t=-1$
$= \frac{1}{2}$

$y - 0 = \frac{1}{2}(x-3)$
$y = \frac{1}{2}x - \frac{3}{2}$ *

HOR. TANGENT $dy/dt = 1+2t = 0 \Rightarrow t = -\frac{1}{2} \Rightarrow \left(\frac{9}{4}, -\frac{1}{4}\right)$

VERT TANGENT $dx/dt = 2t = 0 \Rightarrow t=0 \Rightarrow (2,0)$

Ex2. Find the equation of the **TANGENT LINE** to the parametric curve when $t=1$

$$(x,y) = (t^2+4t,\ 2+1/t)$$

sol: POINT @ $t=1 \Rightarrow (5,3)$

SLOPE $dy/dx = \dfrac{dy/dt}{dx/dt} = \dfrac{-1/t^2}{2t+4}$ @ $t=1$ $= \dfrac{-1}{6} = m$

$$y-y_1 = m(x-x_1)$$

$$y-3 = -1/6(x-5)$$

$$\boxed{y = -1/6\,x + 23/6}$$

Ex3. Find the equation of the **TANGENT LINE** to the parametric curve at the point $(0,0)$

$$(x,y) = (2\sin(t),\ 2t+t^2)$$

$x = 2\sin(t) = 0$

$y = 2t+t^2 = 0$

so $t=0$

sol: $dy/dx = \dfrac{dy/dt}{dx/dt} = \dfrac{2+2t}{2\cos(t)}$ @ $t=0$ $= \dfrac{2}{2} = \boxed{1}$ slope

Pt $= (0,0)$

SLOPE $= 1$

$y-y_1 = m(x-x_1)$

$y = x$

Ex 4: For the following parametric curve, at what points does the **TANGENT LINE** to the curve have slope 1?

$$(x,y) = \left(\frac{8t^3}{3}+5,\ 18t^2-16t+1\right)$$

Sol:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{36t-16}{8t^2} \underset{\text{SET}}{=} 1 \xrightarrow{\text{SOLVE FOR } t}$$

$$8t^2 = 36t - 16$$
$$2t^2 = 9t - 4$$
$$2t^2 - 9t + 4 = 0$$
$$(2t-1)(t-4) = 0$$
$$t = 1/2 \quad t = 4$$

$\downarrow$ find points

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Ex 5. Show that the following parametric curve has two **TANGENT LINES** at the point (0,0) and find equations of both of them.

$$(x,y) = (2\sin(\theta),\ \cos(\theta)\sin(\theta)) \qquad 0 \le t < 2\pi$$

Sol:

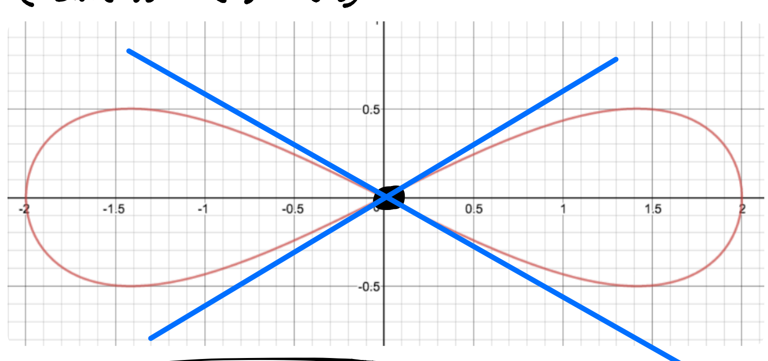
find θ

$0 = 2\sin\theta$

$\theta = 0, \pi, 2\pi, \ldots$

$0 = \cos(\theta)\sin(\theta)$

$\theta = 0, \pi, 2\pi, \ldots$



$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{2\cos\theta}$$

$\theta = 0 \quad\quad \theta = \pi$

$m = 1/2 \quad\quad m = -1/2$

$$y = \tfrac{1}{2}x, \quad y = -\tfrac{1}{2}x$$